

Multivariable Wilson polynomials and degenerate Hecke algebras

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Abstract. We study a rational version of the double affine Hecke algebra associated to the nonreduced affine root system of type (C_n^\vee, C_n) . A certain representation in terms of difference-reflection operators naturally leads to the definition of nonsymmetric versions of the multivariable Wilson polynomials. Using the degenerate Hecke algebra we derive several properties, such as orthogonality relations and quadratic norms, for the nonsymmetric and symmetric multivariable Wilson polynomials.

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1. Introduction

The Macdonald theory of multivariable orthogonal polynomials associated to root systems can be understood using a specific representation of Cherednik's double affine Hecke algebra (DAHA) in terms of q -difference-reflection operators, see e.g. Cherednik [2] and Macdonald [12]. Koornwinder polynomials [11] generalize the Macdonald polynomials associated to classical root systems. The algebraic structure underlying the Koornwinder polynomials is Sahi's DAHA associated to the nonreduced affine root system of type (C_n^\vee, C_n) , see Noumi [14], Sahi [16, 17], and Stokman [18]. The Koornwinder polynomials depend, besides the base q , on five parameters, corresponding to the number of W -orbits in the affine root system of type (C_n^\vee, C_n) with corresponding affine Weyl group W . Many families of multivariable orthogonal polynomials can be obtained as limits of the Koornwinder polynomials, see e.g. [21] and [19].

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Several of these families have been associated in the literature to degenerate versions of double affine Hecke algebras. For instance, the Heckman–Opdam polynomials [10] are naturally associated to a trigonometric degenerate DAHA [15].

In this paper, we study an algebra \mathcal{H} which is a rational degeneration of the DAHA of type (C_n^\vee, C_n) , and we show that multivariable Wilson polynomials are associated to \mathcal{H} in a natural way. Multivariable Wilson polynomials are limits of the Koornwinder polynomials (for $q \rightarrow 1$) depending on five limiting parameters, see van Diejen [20, 21]. The one-variable Wilson polynomials [22] are the most general orthogonal polynomials of hypergeometric type, i.e., all hypergeometric orthogonal polynomials, for instance, the Jacobi polynomials, can be obtained as a limit of the one-variable Wilson polynomials. It may be expected that the multivariable Wilson polynomials play a similar role in the theory of orthogonal polynomials of hypergeometric type associated to root systems. Let us show that Wilson polynomials formally generalize the BC -type Heckman–Opdam polynomials. Under suitable conditions on the parameters the multivariable Wilson polynomials are orthogonal on $(i\mathbb{R}_+)^n$ with respect to the weight function

$$\Delta_+(x) = \prod_{1 \leq j < k \leq n} \frac{\Gamma(t \pm x_j \pm x_k)}{\Gamma(\pm x_j \pm x_k)} \prod_{j=1}^n \frac{\Gamma(a \pm x_j) \Gamma(b \pm x_j) \Gamma(c \pm x_j) \Gamma(d \pm x_j)}{\Gamma(\pm 2x_j)}.$$

Here we use the notation $\Gamma(\alpha \pm \beta) = \Gamma(\alpha + \beta) \Gamma(\alpha - \beta)$. The weight function $\Delta_+(x)$ may be considered as a generalization of the weight function for the BC -type Heckman–Opdam polynomials. Indeed, divide by a factor

$$\prod_{j=1}^n \Gamma(a + c + (j-1)t) \Gamma(b + c + (j-1)t) \\ \times \Gamma(a + d + (j-1)t) \Gamma(b + d + (j-1)t),$$

and substitute

$$a = b = \frac{\alpha + 1}{2}, \quad c = \frac{\beta + 1}{2} + i\gamma, \quad d = \frac{\beta + 1}{2} - i\gamma, \quad x_j = i\gamma\sqrt{y_j},$$

then after applying Stirling's formula we find that in the limit $\gamma \rightarrow \infty$,

$$\Delta_+(x) \sim \frac{\sqrt{y_1 \cdots y_n}}{\gamma^n} \prod_{1 \leq j < k \leq n} |y_j - y_k|^{2t} \prod_{j=1}^n y_j^\alpha (1 - y_j)^\beta, \quad y_1, \dots, y_n \in (0, 1),$$

and $\Delta_+(x)$ vanishes exponentially if $y_j > 1$ for any j .

The rational DAHA \mathcal{H} that we study has a faithful representation in terms of difference-reflection operators, much like the representation of the trigonometric DAHA from [1]. Nonsymmetric versions of the multivariable Wilson polynomials then appear in the representation theory of \mathcal{H} as the polynomial eigenfunctions of the analogues of the Cherednik operators. Methods from double affine Hecke algebras, see e.g. [2, 12], can now be used to

obtain properties of the nonsymmetric multivariable Wilson polynomials, such as orthogonality relations and a duality property. The rank one version of the algebra \mathcal{H} and the corresponding one-variable Wilson polynomials have been studied in [8]. Let us remark that a four-parameter subfamily of the (symmetric) multivariable Wilson polynomials have been obtained by Zhang [23] in the context of degenerate Hecke algebras as the Harish-Chandra transform of the BC -type Heckman–Opdam polynomials. This shows that for suitable parameters the multivariable Wilson polynomials, multiplied by certain Γ -functions, form an orthogonal basis of the L^2 -space consisting of functions on \mathbb{R}^n with the Harish-Chandra measure $|c(\lambda)|^{-2}d\lambda$. It would be interesting to see if Zhang’s results can be extended to the nonsymmetric case.

The algebra \mathcal{H} has appeared earlier in the literature; it is isomorphic to a rational generalized DAHA defined by Etingof et al. [3]. Results in [3] then imply that \mathcal{H} is also closely related to several other algebras appearing in the literature, e.g. \mathcal{H} is the spherical subalgebra of a certain deformed preprojective algebra by Gan and Ginzburg [7]. In this paper we obtain several new properties of the algebra \mathcal{H} . One remarkable property is that it contains a subalgebra H that may be considered as a deformation of $\mathbb{C}[W]$ with deformed braid-relations. To be more precise, as an algebra H is generated by T_0, \dots, T_n , which satisfy the following braid-type relations:

$$\begin{aligned} T_i T_{i+1} T_i T_{i+1} + t T_i T_{i+1} &= T_{i+1} T_i T_{i+1} T_i + t T_{i+1} T_i, \quad i = 0, n-1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad i \in [1, n-2], \\ T_i T_j &= T_j T_i, \quad |i-j| \geq 2. \end{aligned}$$

Here t is some complex parameter. For $t = 0$ the above relations are the usual braid relations for W . Deformations like this, i.e. deformations of a group algebra $\mathbb{C}[G]$, G a Coxeter group, with deformed braid relations, have recently been studied by Etingof and Rains [5, 6].

The paper is organized as follows. After some preliminaries on root systems, we define in Sect. 2 a subalgebra H of the trigonometric DAHA \mathfrak{H} , and it is shown that the subalgebra H is a deformation of the group algebra $\mathbb{C}[W]$. Then in Sect. 3 we define the rational DAHA \mathcal{H} as a subalgebra of \mathfrak{H} that is generated by H and a polynomial algebra. We derive several useful properties of \mathcal{H} , such as a PBW-theorem, and we define the analogues of Cherednik’s Y -operators. In Sect. 4 we study a faithful representation π of \mathcal{H} in terms of difference-reflection operators on the vector space \mathcal{P} consisting of polynomials in n variables. The nonsymmetric multivariable Wilson polynomials are used to describe the decomposition of \mathcal{P} into irreducible $\pi(H)$ -modules. In Sect. 5 we derive several properties of the nonsymmetric Wilson polynomials. Finally, in Sect. 6 we show how well-known properties of the (symmetric) multivariable Wilson polynomials can be obtained from the representation theory of \mathcal{H} .

2. The degenerate affine Hecke algebra

In this section we define an algebra H that we consider as a degenerate affine Hecke algebra of type \tilde{C}_n . Throughout the paper we use the following notation: for integers m, n with $m < n$ we write $[m, n]$ for the set $\{m, m+1, \dots, n\}$.

2.1. The affine root system (C_n^\vee, C_n)

We fix an integer $n \geq 2$. Let $\{\epsilon_j\}_{j=1}^n$ be the standard orthonormal basis for $V = \mathbb{R}^n$ with standard inner product $\langle \cdot, \cdot \rangle$. Let \hat{V} denote the vector space of affine linear transformations from V to \mathbb{R} . We identify \hat{V} with $V \oplus \mathbb{R}\delta$ by writing $f \in \hat{V}$ as

$$f(u) = \langle v, u \rangle + c\delta(u), \quad u \in V,$$

for some $v \in V$ and $c \in \mathbb{R}$, where $\delta : V \rightarrow \mathbb{R}$ is the constant function defined by $\delta(u) = 1$ for all $u \in V$. The inner product on V is extended to a bilinear form on \hat{V} by

$$\langle \lambda_1 + \mu_1\delta, \lambda_2 + \mu_2\delta \rangle = \langle \lambda_1, \lambda_2 \rangle, \quad \lambda_1, \lambda_2 \in V, \quad \mu_1, \mu_2 \in \mathbb{R}.$$

We define

$$a_0 = \delta - 2\epsilon_1, \quad a_i = \epsilon_i - \epsilon_{i+1} \quad (i = 1, \dots, n-1), \quad a_n = 2\epsilon_n.$$

The elements $a_i \in \hat{V}$ are the simple roots for the affine root system of type \tilde{C}_n . Let W be the subgroup of $\text{GL}(\hat{V})$ generated by the simple reflections $s_{a_i} = s_i$, $i = 0, \dots, n$, where for $\beta \in \hat{V}$ with $\langle \beta, \beta \rangle \neq 0$ the reflection s_β is defined by

$$s_\beta f = f - \langle f, \beta^\vee \rangle \beta, \quad \beta^\vee = \frac{2\beta}{\langle \beta, \beta \rangle},$$

for $f \in \hat{V}$. The group W is a Coxeter group with relations $s_i^2 = 1$, $i \in [0, n]$, and

$$\begin{aligned} s_i s_{i+1} s_i s_{i+1} &= s_{i+1} s_i s_{i+1} s_i, \quad i = 0, n-1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \quad i \in [1, n-2], \\ s_i s_j &= s_j s_i, \quad i, j \in [0, n], \quad |i-j| \geq 2. \end{aligned}$$

The set $\mathcal{R}_r = W\{a_0, \dots, a_n\} \subset \hat{V}$ is the (reduced) affine root system of type \tilde{C}_n . The nonreduced affine root system of type (C_n^\vee, C_n) is the root system $\mathcal{R} = \mathcal{R}_r^\vee \cup \mathcal{R}_r$. The root systems $\mathcal{R}, \mathcal{R}_r^\vee$ and \mathcal{R}_r all have W as corresponding affine Weyl group. We denote by $W_0 \subset W$ the (finite) Weyl group generated by s_1, \dots, s_n . Note that the subgroup generated by s_1, \dots, s_{n-1} is isomorphic to the symmetric group \mathcal{S}_n . The set $\Sigma = W_0\{a_1, \dots, a_n\}$ is the root system of type C_n . We write $\mathcal{R}^\pm, \mathcal{R}_r^\pm, \Sigma^\pm$ for the positive/negative roots in $\mathcal{R}, \mathcal{R}_r, \Sigma$, respectively, with respect to our choice of simple roots. In particular,

$$\mathcal{R}_r^+ = \Sigma^+ \cup \{\alpha \in \mathcal{R}_r \mid \alpha(0) > 0\}.$$

The co-root lattice Q^\vee and the weight lattice P of Σ in V can both be identified with $\Lambda = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$. We denote by Λ^+ the cone of dominant weights, i.e., $\Lambda^+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i$, where $\omega_i = \epsilon_1 + \cdots + \epsilon_i$. There is an alternative description of the affine Weyl group W as the semidirect product

$$W = W_0 \ltimes \tau(\Lambda),$$

where $\tau(\lambda)$, $\lambda \in \Lambda$, is the translation operator given by $\tau(\lambda)f = f + \langle \lambda, f \rangle \delta$ for $f \in \widehat{V}$. The translation operator $\tau(\epsilon_i)$, $i \in [1, n]$, can be expressed in terms of the simple reflections by

$$\tau(\epsilon_i) = s_i \cdots s_{n-1} s_n s_{n-1} \cdots s_1 s_0 s_1 \cdots s_{i-1}. \quad (2.1)$$

This is a reduced expression.

2.2. A degenerate affine Hecke algebra

Let $V_{\mathbb{C}} = V \oplus iV$ be the complexification of V . The finite Weyl group W_0 acts on $V_{\mathbb{C}}$ by reflections as defined in the previous subsection. We get a representation of the affine Weyl group W on $V_{\mathbb{C}}$ by defining $\tau(\lambda)x = x - \lambda$ for $\lambda \in \Lambda$ and $x \in V_{\mathbb{C}}$. Let \mathcal{P} be the algebra consisting of polynomials on $V_{\mathbb{C}}$, then W acts on \mathcal{P} by $(wp)(x) = p(w^{-1}x)$, in particular

$$\begin{aligned} (s_0 p)(x) &= p(1 - x_1, x_2, \dots, x_n), \\ (s_i p)(x) &= p(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n), \quad i \in [1, n-1], \\ (s_n p)(x) &= p(x_1, \dots, x_{n-1}, -x_n), \end{aligned}$$

where $x = (x_1, \dots, x_n)$. We denote by \mathcal{P}^{W_0} the algebra consisting of polynomials $p \in \mathcal{P}$ that are invariant under the action of the finite Weyl group W_0 , i.e., $s_i p = p$ for $i \in [1, n]$. The set of monomials $\{x^\mu\}_{\mu \in \mathbb{Z}_{\geq 0}^n}$ forms a linear basis for \mathcal{P} . Here we use the notation $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$ for $x = (x_1, \dots, x_n)$ and $\mu_i = \langle \mu, \epsilon_i \rangle$. It will be convenient to label this basis with elements in Λ . For this we define a bijection $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\phi(m) = \begin{cases} 2m, & m \geq 0, \\ -2m - 1, & m < 0. \end{cases}$$

Now for $\lambda = \sum \lambda_i \epsilon_i \in \Lambda$ we denote

$$\phi(\lambda) = \sum_{i=1}^n \phi(\lambda_i) \epsilon_i.$$

Then the set $\{x^{\phi(\lambda)}\}_{\lambda \in \Lambda}$ is a linear basis for \mathcal{P} .

Let $\mathbf{k} : \mathcal{R}_r \rightarrow \mathbb{C}$ be a multiplicity function on \mathcal{R}_r , i.e., \mathbf{k} is constant on W -orbits in \mathcal{R}_r . The function \mathbf{k} is uniquely determined by its values on a_0, a_1 and a_n , so we may identify \mathbf{k} with the ordered 3-tuple (k_0, k_1, k_n) , where $k_i = \mathbf{k}(a_i)$. Let \mathcal{P}_X be the algebra $\mathbb{C}[X_1, \dots, X_n]$. If $p(x) = \sum_{\mu} c_{\mu} x^{\mu} \in \mathcal{P}$ is a polynomial, we denote by $p(X)$ the element $\sum_{\mu} c_{\mu} X^{\mu}$ in \mathcal{P}_X , where $X^{\mu} = X_1^{\mu_1} \cdots X_n^{\mu_n}$ for $\mu_i = \langle \mu, \epsilon_i \rangle$. The degenerate (trigonometric) DAHA

$\mathfrak{H}(\mathbf{k})$ associated to the root system \mathcal{R}_r is the complex associative algebra generated by W (with generators r_i , $i = 0, \dots, n$) and \mathcal{P}_X , with the cross-relations

$$r_i p(X) - (s_i p)(X) r_i = \frac{k_i}{a_i(X)} ((s_i p)(X) - p(X)), \quad i \in [0, n].$$

Here we denote the generators of W considered as elements of $\mathfrak{H}(\mathbf{k})$ by r_i to distinguish them from the operators s_i . If we say that an algebra A is generated by the algebras A' and A'' (both having a unit element) we mean the following:

- $A \simeq A' \otimes A''$ as vector spaces,
- for $a' \in A'$ and $a'' \in A''$ the maps $a' \mapsto a' \otimes 1$ and $a'' \mapsto 1 \otimes a''$ are algebra homomorphisms,
- $(a' \otimes 1)(1 \otimes a'') = a' \otimes a''$, for $a' \in A'$, $a'' \in A''$.

Moreover, we will write $a'a''$ instead of $a' \otimes a''$.

The algebra $\mathfrak{H}(\mathbf{k})$ has a linear basis, called the Poincaré–Birkhoff–Witt (PBW) basis,

$$\left\{ wX^{\phi(\lambda)} \mid w \in W, \lambda \in \Lambda \right\}.$$

Let $\mathbf{t} : \mathcal{R} \rightarrow \mathbb{C}$ be a multiplicity function on \mathcal{R} , that we identify with the ordered 5-tuple $(\mathbf{t}(a_0), \mathbf{t}(a_0^\vee), \mathbf{t}(a_1), \mathbf{t}(a_n), \mathbf{t}(a_n^\vee))$. To simplify notation, we write

$$\mathbf{t} = (t_0, u_0, t, t_n, u_n), \quad (2.2)$$

and we also use the notations

$$\mathbf{t}_{\mathcal{R}_r} = (t_0, t, t_n) \quad \text{and} \quad \mathbf{t}_{\mathcal{R}_r^\vee} = (u_0, t, u_n). \quad (2.3)$$

We assume that the function \mathbf{t} is related to the multiplicity function \mathbf{k} by

$$k_0 = 2t_0 + 2u_0, \quad k_1 = t, \quad k_n = 2t_n + 2u_n.$$

We now define a subalgebra $H(\mathbf{t}_{\mathcal{R}_r})$ of $\mathfrak{H}(\mathbf{k})$ that may be considered as a rational version of the affine Hecke algebra of type \tilde{C}_n .

Definition 2.1. The algebra $H = H(\mathbf{t}_{\mathcal{R}_r}) \subset \mathfrak{H}(\mathbf{k})$ is the subalgebra generated by r_1, \dots, r_{n-1} and

$$T_i = t_i + \frac{1}{2}(t_i - u_i + a_i^\vee(X))(r_i - 1), \quad i = 0, n.$$

Although it is not clear at this point, the algebra H is an (u_0, u_n) -dependent subalgebra of $\mathfrak{H}(\mathbf{k})$ that depends as an abstract algebra only on $\mathbf{t}_{\mathcal{R}_r} = (t_0, t, t_n)$. When working in the algebra H we will write $T_i = r_i$ for $i \in [1, n-1]$, so H is the algebra generated by T_i , $i \in [0, n]$. We also define the algebra $H_0 \subset H$ to be the subalgebra generated by T_i , $i \in [1, n]$. This is the analogue of the finite Hecke algebra of type C_n .

Proposition 2.2. *We have the following relations in the algebra H :*

Quadratic relations:

$$\begin{aligned} T_i^2 &= t_i^2, \quad i = 0, n, \\ T_i^2 &= 1, \quad i \in [1, n-1], \end{aligned}$$

Braid-type relations:

$$\begin{aligned} T_i T_{i+1} T_i T_{i+1} + t T_i T_{i+1} &= T_{i+1} T_i T_{i+1} T_i + t T_{i+1} T_i, \quad i = 0, n-1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad i \in [1, n-2], \\ T_i T_j &= T_j T_i, \quad i, j \in [0, n], \quad |i-j| \geq 2. \end{aligned}$$

Proof. We only need to verify the relations involving T_0 and T_n , the other ones follow directly from the definition of the algebra H . For the quadratic relations we find from the definition of T_i , $i = 0, n$, and the relations in $\mathfrak{H}(\mathbf{k})$,

$$\begin{aligned} [r_i - 1][t_i - u_i + a_i^\vee(X)][r_i - 1] \\ = [(t_i - u_i - a_i^\vee(X))r_i - 3t_i - u_i - a_i^\vee(X)][r_i - 1] \\ = -4t_i(r_i - 1) \end{aligned}$$

which gives

$$\begin{aligned} T_i^2 &= t_i^2 + t_i[t_i - u_i + a_i^\vee(X)][r_i - 1] \\ &\quad + \frac{1}{4}[t_i - u_i + a_i^\vee(X)][r_i - 1][t_i - u_i + a_i^\vee(X)][r_i - 1] \\ &= t_i^2. \end{aligned}$$

Next we check the braid-type relation for T_n . The braid-type relation for T_0 is checked in the same way. Let us write

$$f(y) = \frac{1}{2}(t_n - u_n - y), \quad g(y) = \frac{1}{2}(t_n + u_n + y),$$

then it follows from the relations in $\mathfrak{H}(\mathbf{k})$ that

$$T_n = t_n + (s_n f)(X_n)(r_n - 1) = r_n f(X_n) - g(X_n).$$

Note that we also have

$$\begin{aligned} f(X_n)r_{n-1} &= r_{n-1}f(X_{n-1}) - t/2, \\ g(X_n)r_{n-1} &= r_{n-1}g(X_{n-1}) + t/2. \end{aligned}$$

Now we find in $\mathfrak{H}(\mathbf{k})$

$$T_n r_{n-1} = r_n r_{n-1} f(X_{n-1}) - \frac{1}{2} t r_n - r_{n-1} g(X_{n-1}) - t/2.$$

Multiplying this from the right by $T_n = r_n f(X_n) - g(X_n)$ we obtain, after some calculations

$$\begin{aligned} T_n r_{n-1} T_n + t T_n &= r_n r_{n-1} r_n f(X_{n-1}) f(X_n) - r_n r_{n-1} f(X_{n-1}) g(X_n) \\ &\quad - r_{n-1} r_n g(X_{n-1}) f(X_n) + r_{n-1} g(X_{n-1}) g(X_n) \\ &\quad + \frac{1}{2} t (r_n - 1) [f(X_n) + g(X_n)]. \end{aligned} \tag{2.4}$$

Note that $f(y) + g(y) = t_n$. Now we multiply (2.4) from the right by r_{n-1} and bring r_{n-1} to the left of f and g , then we have after a few calculations

$$\begin{aligned} T_n r_{n-1} T_n r_{n-1} + t T_n r_{n-1} \\ = r_n r_{n-1} r_n r_{n-1} f(X_n) f(X_{n-1}) - r_{n-1} r_n r_{n-1} g(X_n) f(X_{n-1}) \\ + \frac{1}{2} t_n t r_{n-1} r_n - r_n f(X_n) g(X_{n-1}) - \frac{1}{2} t_n t r_{n-1} + g(X_n) g(X_{n-1}). \end{aligned}$$

By inspection, using the braid relations for r_n , this is the same as (2.4) multiplied by r_{n-1} from the left. \square

The previous proposition shows that H can be considered as a deformation of the group algebra $\mathbb{C}[W]$, with deformed braid-relations for $i = 0, n$.

In order to show that the relations from Proposition 2.2 characterize H as an algebra we show that H has a PBW-basis. We introduce the following notation. If u is a word in the s_i 's, $u = s_{i_1} \cdots s_{i_r}$, then we write $T_u = T_{i_1} \cdots T_{i_r}$.

Proposition 2.3 (PBW-property for H). *For every $w \in W$ let \bar{w} be a fixed reduced expression for w , then the set $\{T_{\bar{w}} \mid w \in W\}$ is a linear basis for H .*

Proof. The set $\{T_{\bar{w}} \mid w \in W\}$ spans H , see [5, Theorem 2.3]. Now suppose that we have a relation $\sum_{u \in W} c_u T_{\bar{u}} = 0$ in H for some coefficients $c_u \in \mathbb{C}$. Using the cross-relations in $\mathfrak{H}(\mathbf{k})$, we may write

$$T_{\bar{u}} = \sum_{\substack{w \in W \\ w \leq u}} f_{u,w}(X) w,$$

where $f_{u,w}(X) \in \mathcal{P}_X$ and $f_{u,u}$ is nonzero. Here $w \leq u$ is meant with respect to the Bruhat order on W . Now we have

$$\sum_{\substack{u, w \in W \\ w \leq u}} c_u f_{u,w}(X) w = 0,$$

which implies, by the PBW-property for \mathfrak{H} , that for any $w \in W$

$$\sum_{u \geq w} c_u f_{u,w}(X) = 0.$$

Let v be a maximal element (in the Bruhat order) in the set $\{u \in W \mid c_u \neq 0\}$, then $c_v f_{v,v}(X) = 0$ in \mathcal{P}_X . But since $f_{v,v}$ is a nonzero polynomial, it follows that $c_v = 0$. We conclude that $c_u = 0$ for all $u \in W$. \square

Corollary 2.4. *As an algebra H is completely characterized by the relations from Proposition 2.2.*

Proof. Suppose that we have a complex associative algebra \mathcal{V} generated by V_i , $i = 0, \dots, n$, with the same relations as in Proposition 2.2 (with T_i replaced by V_i). Then the assignments $V_i \mapsto T_i$ extend to a surjective homomorphism $\psi : \mathcal{V} \rightarrow H$. Now let $V \in \mathcal{V}$ satisfy $\psi(V) = 0$. By [5, Theorem 2.3] the set $\{V_{\bar{w}} \mid w \in W\}$ spans \mathcal{V} . Writing $V = \sum_{w \in W} c_w V_{\bar{w}}$ and applying ψ shows that

$\sum_{w \in W} c_w T_{\bar{w}} = 0$, hence every c_w is equal to zero by Proposition 2.3. This shows that ψ is also injective. \square

We also have the PBW-property for the finite algebra H_0 .

Corollary 2.5. *The set $\{T_{\bar{w}} \mid w \in W_0\}$ is a linear basis for H_0 .*

From Proposition 2.2 and Corollary 2.4 it is now clear that H depends as an abstract algebra only on the parameters t_0, t, t_n . Naturally we also have an algebra $H(\mathbf{t}_{\mathcal{R}_r^\vee})$ related to the reduced root system \mathcal{R}_r^\vee which is obtained from the algebra $H(\mathbf{t}_{\mathcal{R}_r})$ by replacing (t_0, t_n) by (u_0, u_n) .

3. The rational double affine Hecke algebra

We now come to the definition of the algebra $\mathcal{H}(\mathbf{t})$ that may be considered as a rational degeneration of Sahi's double affine Hecke algebra for type (C^\vee, C) . This is explained in Remark 4.2. We call this algebra the rational double affine Hecke algebra. For rank 1 this algebra is studied in [8]. Let us remark that \mathcal{H} is not a rational Cherednik algebra as defined in [4]. We show in Sect. 3.4 that the algebra $\mathcal{H}(\mathbf{t})$ is isomorphic to the rational generalized double affine Hecke algebra attached to an affine Dynkin diagram of type \tilde{D}_4 as defined by Etingof et al. [3].

Definition 3.1. The algebra $\mathcal{H} = \mathcal{H}(\mathbf{t}) \subset \mathfrak{H}(\mathbf{k})$ is the subalgebra generated by $H(\mathbf{t}_{\mathcal{R}_r})$ and \mathcal{P}_X .

From the relations in $\mathfrak{H}(\mathbf{k})$ it follows that in \mathcal{H} we have, for $p \in \mathcal{P}$,

$$T_i p(X) - (s_i p)(X) T_i = d_i(X; \mathbf{t}) [(s_i p)(X) - p(X)], \quad i \in [0, n], \quad (3.1)$$

where

$$d_i(X; \mathbf{t}) = \begin{cases} \frac{t_i^2 - u_i^2 + a_i^\vee(X)^2}{a_i(X)}, & i = 0, n, \\ \frac{t}{a_i(X)}, & i \in [1, n-1]. \end{cases}$$

Together with the PBW-Theorem for the algebra H this leads to the PBW-Theorem for \mathcal{H} .

Proposition 3.2 (PBW property for \mathcal{H}). *The sets*

$$\{X^{\phi(\lambda)} T_{\bar{w}} \mid \lambda \in \Lambda, w \in W\}, \quad \{T_{\bar{w}} X^{\phi(\lambda)} \mid \lambda \in \Lambda, w \in W\}$$

form linear bases for \mathcal{H} .

Corollary 3.3. *The algebra \mathcal{H} is completely characterized as an algebra by the relations for H from Proposition 2.2, the relations $X_i X_j = X_j X_i$ for $i, j \in [1, n]$, and the cross relations (3.1).*

We will frequently use (3.1) with $p(X) = X_i$, $i \in [1, n]$. In fact, (3.1) can be recovered from these relations. We collect the identities in the following lemma.

Lemma 3.4. *In \mathcal{H} the following relations hold:*

$$\begin{aligned} (X_1 - \frac{1}{2} - T_0)^2 &= u_0^2, \\ (X_n + T_n)^2 &= u_n^2, \\ T_i X_i T_i &= X_{i+1} - t T_i, \quad i \in [1, n-1], \\ X_i T_j &= T_j X_i, \quad i \in [1, n], \quad j \in [0, n], \quad |i-j| \geq 2, \\ X_i T_{i+1} &= T_{i+1} X_i, \quad i \in [1, n-1]. \end{aligned}$$

We define in \mathcal{H}

$$T_0^\vee = X_1 - \frac{1}{2} - T_0, \quad T_n^\vee = -X_n - T_n,$$

then we have $(T_0^\vee)^2 = u_0^2$ and $(T_n^\vee)^2 = u_n^2$. For $i \in [1, n-1]$ we will sometimes denote $T_i^\vee = T_i$. Let us introduce some convenient notations. We write for $i, j \in [1, n]$, $i < j$,

$$\begin{aligned} T_{i,j} &= T_i T_{i+1} \cdots T_{j-2} T_{j-1} T_{j-2} \cdots T_{i+1} T_i \\ &= T_{j-1} T_{j-2} \cdots T_{i+1} T_i T_{i+1} \cdots T_{j-2} T_{j-1}. \end{aligned}$$

This notation corresponds to the familiar notation in the symmetric group \mathcal{S}_n where s_{ij} denotes the transposition $i \leftrightarrow j$. Recall here that the elements T_i , $i = 1, \dots, n-1$, generate a subalgebra isomorphic to \mathcal{S}_n . We also define in \mathcal{H}

$$\begin{aligned} \Xi_{i,n} &= T_i T_{i+1} \cdots T_{n-1} T_n T_{n-1} \cdots T_{i+1} T_i, \\ \Xi_{i,n}^\vee &= T_i T_{i+1} \cdots T_{n-1} T_n^\vee T_{n-1} \cdots T_{i+1} T_i, \\ \Xi_{0,i} &= T_{i-1} T_{i-2} \cdots T_1 T_0 T_1 \cdots T_{i-2} T_{i-1}, \\ \Xi_{0,i}^\vee &= T_{i-1} T_{i-2} \cdots T_1 T_0^\vee T_1 \cdots T_{i-2} T_{i-1}, \end{aligned}$$

for $i \in [1, n]$.

We give a few useful relations in \mathcal{H} involving T_0^\vee and T_n^\vee .

Lemma 3.5. *The following relations hold in \mathcal{H} :*

$$T_0 T_1 T_0^\vee T_1 = T_1 T_0^\vee T_1 T_0 \quad (3.2a)$$

$$T_n T_{n-1} T_n^\vee T_{n-1} = T_{n-1} T_n^\vee T_{n-1} T_n \quad (3.2b)$$

$$T_0^\vee X_i = X_i T_0^\vee, \quad i \in [2, n] \quad (3.2c)$$

$$T_n^\vee X_i = X_i T_n^\vee, \quad i \in [1, n-1] \quad (3.2d)$$

$$T_0 T_n^\vee = T_n^\vee T_0 \quad (3.2e)$$

$$T_0^\vee T_n = T_n T_0^\vee \quad (3.2f)$$

$$T_0^\vee T_n^\vee = T_n^\vee T_0^\vee \quad (3.2g)$$

$$\frac{1}{2} + T_0 + T_0^\vee + \Xi_{1,n} + \Xi_{1,n}^\vee + t \sum_{j=2}^n T_{1,j} = 0. \quad (3.2h)$$

Proof. Relations (3.2c)–(3.2g) are obvious. For (3.2b) we use the relations

$$T_{n-1}X_nT_{n-1} = X_{n-1} + tT_{n-1}, \quad T_nX_{n-1} = X_{n-1}T_n$$

from Lemma 3.4 and the braid-type relations for T_n ;

$$\begin{aligned} T_nT_{n-1}(X_n + T_n)T_{n-1} \\ &= T_n(X_{n-1} + tT_{n-1}) + T_{n-1}T_nT_{n-1}T_n + tT_{n-1}T_n - tT_nT_{n-1} \\ &= (X_{n-1} + tT_{n-1})T_n + T_{n-1}T_nT_{n-1}T_n \\ &= T_{n-1}(X_n + T_n)T_{n-1}T_n. \end{aligned}$$

Relation (3.2a) follows in the same way. In order to prove (3.2h), we observe that $T_iX_{i+1}T_i = X_i + tT_i$ for $i \in [1, n-1]$, so that by backward induction

$$T_1 \cdots T_{n-1}X_nT_{n-1} \cdots T_1 = X_1 + t \sum_{j=2}^n T_{1,j}.$$

Then (3.2h) follows from the identities $T_0^\vee + T_0 + \frac{1}{2} = X_1$ and $T_n + T_n^\vee = -X_n$, see (2.3). \square

Next we show that \mathcal{H} has a subalgebra isomorphic to the degenerate affine Hecke algebra $H(\mathfrak{t}_{\mathcal{R}_r^\vee})$.

Proposition 3.6. *The subalgebra of \mathcal{H} generated by T_j^\vee , $j = 0, \dots, n$, is isomorphic to $H(\mathfrak{t}_{\mathcal{R}_r^\vee})$, i.e., it is characterized by the following relations:*

Quadratic relations:

$$\begin{aligned} (T_i^\vee)^2 &= u_i^2, \quad i = 0, n, \\ (T_i^\vee)^2 &= 1, \quad i \in [1, n-1]. \end{aligned}$$

Braid-type relations:

$$\begin{aligned} T_i^\vee T_{i+1}^\vee T_i^\vee T_{i+1}^\vee + tT_i^\vee T_{i+1}^\vee &= T_{i+1}^\vee T_i^\vee T_{i+1}^\vee T_i^\vee + tT_{i+1}^\vee T_j^\vee, \quad i = 0, n-1, \\ T_i^\vee T_{i+1}^\vee T_i^\vee &= T_{i+1}^\vee T_i^\vee T_{i+1}^\vee, \quad i \in [1, n-2], \\ T_i^\vee T_j^\vee &= T_j^\vee T_i^\vee, \quad i, j \in [0, n], \quad |i-j| \geq 2. \end{aligned}$$

Proof. Let H^\vee be the subalgebra generated by T_i^\vee , $i = 0, \dots, n$. Let us first verify that the relations mentioned in the proposition are satisfied in H^\vee . We only need to check the braid-type relations involving T_0^\vee and T_n^\vee . Let us concentrate on the relations for T_n^\vee . We already know from Lemma 3.5 that $T_iT_n^\vee = T_n^\vee T_i$ for $i \in [1, n-2]$. The relation involving T_{n-1} follows from writing out $X_nX_{n-1} = X_{n-1}X_n$ if we write $X_{n-1} = T_{n-1}X_nT_{n-1} + tT_{n-1}$, $X_n = -T_n - T_n^\vee$, and we use the braid-type relation for T_n and (3.2b). For T_0^\vee the proof is similar.

In order to show that the H^\vee is indeed isomorphic to $H(\mathfrak{t}_{\mathcal{R}_r^\vee})$ it is now enough to show that $\{T_w^\vee \mid w \in W\}$ is a PBW-basis for H^\vee . This is done in the same way as in Proposition 2.3. \square

We now come to the following characterization of the rational DAHA \mathcal{H} , which is the analogue of [18, Theorem 3.4].

Theorem 3.7. *The algebra \mathcal{H} is isomorphic to the complex associative algebra \mathcal{V} generated by $V_0, V_0^\vee, V_n, V_n^\vee, V_i = V_i^\vee, i = 1, \dots, n-1$, such that*

- (i) *the subalgebra generated by V_i ($i = 0, \dots, n$) is isomorphic to $H(\mathfrak{t}_{\mathcal{R}_r})$,*
- (ii) *the subalgebra generated by V_i^\vee ($i = 0, \dots, n$) is isomorphic to $H(\mathfrak{t}_{\mathcal{R}_r^\vee})$,*
- (iii) *the following compatibility relations are satisfied:*

$$\begin{aligned} [V_0^\vee, V_n] &= 0, \quad [V_n^\vee, V_0] = 0, \\ \frac{1}{2} + V_0 + V_0^\vee + \Upsilon_{1,n} + \Upsilon_{1,n}^\vee + t \sum_{j=2}^n V_{1,j} &= 0, \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} V_{1,j} &= V_1 V_2 \cdots V_{j-2} V_{j-1} V_{j-2} \cdots V_2 V_1, \quad j \in [2, n], \\ \Upsilon_{1,n} &= V_1 V_2 \cdots V_{n-1} V_n V_{n-1} \cdots V_2 V_1, \\ \Upsilon_{1,n}^\vee &= V_1 V_2 \cdots V_{n-1} V_n^\vee V_{n-1} \cdots V_2 V_1. \end{aligned}$$

The isomorphism $\varphi : \mathcal{V} \rightarrow \mathcal{H}$ is given on the generators of \mathcal{V} by

$$\varphi(V_i) = T_i, \quad \varphi(V_i^\vee) = T_i^\vee,$$

for $i \in [0, n]$.

We prove the theorem in the next subsection.

3.1. Proof of Theorem 3.7

First note that φ preserves the defining relations for \mathcal{V} , and φ maps generators of \mathcal{V} to generators of \mathcal{H} , hence φ is a surjective algebra homomorphism. In order to prove that φ is also injective, we show that we can recover the defining relations for \mathcal{H} in terms of $T_0, \dots, T_n, X_1, \dots, X_n$, see Corollary 3.3, from the relations in \mathcal{V} . Since the elements $V_i, i = 0, \dots, n$, generate an algebra isomorphic to $H(\mathfrak{t}_{\mathcal{R}_r})$, we need to find pairwise commuting elements $v_i \in \mathcal{V}, i = 1, \dots, n$, such that $\varphi(v_i) = X_i$, and the following cross-relations holds, see (3.1):

$$V_i p(v) - (s_i p)(v) V_i = d_i(v; \mathfrak{t}) [(s_i p)(v) - p(v)], \quad i \in [0, n]. \quad (3.4)$$

Here we use the usual notation $p(v) = \sum_\mu c_\mu v^\mu$, if $p(x)$ is the polynomial $\sum_\mu c_\mu x^\mu$, and $v^\mu = v_1^{\mu_1} \cdots v_n^{\mu_n}$ for $\mu \in \mathbb{Z}_{\geq 0}^n$.

Let us first introduce the elements v_i . We define in \mathcal{V}

$$v_n = -V_n - V_n^\vee,$$

and we define v_i recursively by

$$v_i = V_i v_{i+1} V_i - t V_i, \quad i \in [1, n-1], \quad (3.5)$$

then $\varphi(v_i) = X_i$ for $i \in [1, n]$. Note that we may write v_i explicitly as

$$v_i = - \left(\Upsilon_{i,n} + \Upsilon_{i,n}^\vee + t \sum_{j=i+1}^n V_{i,j} \right), \quad i \in [1, n], \quad (3.6)$$

where

$$\begin{aligned} V_{i,j} &= V_i V_{i+1} \cdots V_{j-2} V_{j-1} V_{j-2} \cdots V_{i+1} V_i, \quad 1 \leq i < j \leq n, \\ \Upsilon_{i,n} &= V_i V_{i+1} \cdots V_{n-1} V_n V_{n-1} \cdots V_{i+1} V_i, \quad i \in [1, n], \\ \Upsilon_{i,n}^\vee &= V_i V_{i+1} \cdots V_{n-1} V_n^\vee V_{n-1} \cdots V_{i+1} V_i, \quad i \in [1, n]. \end{aligned}$$

For v_1 the compatibility relation (3.3) gives

$$v_1 = \frac{1}{2} + V_0 + V_0^\vee,$$

so using the recursion relation for v_i again we also have

$$v_i = \frac{1}{2} + \Upsilon_{0,i} + \Upsilon_{0,i}^\vee + t \sum_{j=2}^i V_{1,j}, \quad i \in [1, n], \quad (3.7)$$

where

$$\begin{aligned} \Upsilon_{0,i} &= V_{i-1} V_{i-2} \cdots V_1 V_0 V_1 \cdots V_{i-2} V_{i-1}, \quad i \in [1, n], \\ \Upsilon_{0,i}^\vee &= V_{i-1} V_{i-2} \cdots V_1 V_0^\vee V_1 \cdots V_{i-2} V_{i-1}, \quad i \in [1, n]. \end{aligned}$$

We can now show that we have all the relations from Lemma 3.4 in \mathcal{V} .

Lemma 3.8. *The following relations hold in \mathcal{V} :*

$$\begin{aligned} \left(v_1 - \frac{1}{2} - V_0 \right)^2 &= u_0^2, \\ (v_n + V_n)^2 &= u_n^2, \\ V_i v_i V_i &= v_{i+1} - t V_i, \quad i \in [1, n-1], \\ v_i V_j &= V_j v_i, \quad j \in [0, n], \quad i \in [1, n], \quad |i-j| \geq 2, \\ v_i V_{i+1} &= V_{i+1} v_i, \quad i \in [1, n-1]. \end{aligned}$$

Proof. The first three relations follow directly from the definition of v_i . Fix $i \in [2, n]$. In \mathcal{V} we have

$$\begin{aligned} [U_1, U_2] &= 0, \quad U_1 \in \{V_0^\vee, V_0, V_1, \dots, V_{i-2}\}, \\ U_2 &\in \{V_{i,i+1}, V_{i,i+2}, \dots, \Upsilon_{i,n}, \Upsilon_{i,n}^\vee\}. \end{aligned}$$

These relations are easily verified using $[V_k, V_l] = [V_0^\vee, V_l] = [V_k, V_n^\vee] = 0$ for $k \in [0, j]$, $l \in [i+1, n]$. Now from (3.6) we find for $j \in [0, i-2]$

$$\begin{aligned} V_j v_i &= - \left(V_j \Upsilon_{i,n} + V_j \Upsilon_{i,n}^\vee + t \sum_{k=i+1}^n V_j V_{i,k} \right) \\ &= - \left(\Upsilon_{i,n} V_j + \Upsilon_{i,n}^\vee V_j + t \sum_{k=i+1}^n V_{i,k} V_j \right) \\ &= v_i V_j. \end{aligned}$$

Next we fix $i \in [1, n-1]$. Now we use the relations

$$\begin{aligned} [U_1, U_2] &= 0, \quad U_1 \in \{V_{i+1}, V_{i+2}, \dots, V_n, V_n^\vee\}, \\ U_2 &\in \{\Upsilon_{0,i}, \Upsilon_{0,i}^\vee, V_{1,2}, V_{1,3}, \dots, V_{1,i}\}, \end{aligned}$$

then by (3.7) we have for $j \in [i+1, n]$

$$\begin{aligned} V_j v_i &= \frac{1}{2} V_j + V_j \Upsilon_{0,i} + V_j \Upsilon_{0,i}^\vee + t \sum_{k=2}^i V_j V_{1,k} \\ &= \frac{1}{2} V_j + \Upsilon_{0,i} V_j + \Upsilon_{0,i}^\vee V_j + t \sum_{k=2}^i V_{1,k} V_j \\ &= v_i V_j. \end{aligned}$$

□

We still need to show that the subalgebra generated by v_i , $i = 1, \dots, n$, is commutative. Once the commutativity of the v_i 's is established, the cross-relations (3.4) follow from Lemma 3.8. The following lemma will be useful for proving the commutativity of the v_i 's.

Lemma 3.9. *We have the following relations in \mathcal{V} :*

$$\begin{aligned} v_i V_0^\vee &= V_0^\vee v_i, \quad i \in [2, n], \\ v_i V_n^\vee &= V_n^\vee v_i, \quad i \in [1, n-1], \\ V_n V_{n-1} V_n^\vee V_{n-1} &= V_{n-1} V_n^\vee V_{n-1} V_n \end{aligned}$$

Proof. For the first two relations see the proof of Lemma 3.8. The last relation follows from writing out $[v_{n-1}, V_n] = 0$ in terms of V_{n-1} , V_n and V_n^\vee , and using the braid-type relations for V_n and V_{n-1} . □

We are now ready to prove the commutativity of the v_i 's, which completes the proof of Theorem 3.7.

Lemma 3.10. *For $i, j \in [1, n]$, we have $v_i v_j = v_j v_i$.*

Proof. We first show that $v_{n-1}v_n = v_nv_{n-1}$. By (3.5) and the definition of v_n we have

$$\begin{aligned} v_{n-1}v_n &= V_{n-1}v_nV_{n-1}v_n - tV_{n-1}v_n \\ &= V_{n-1}(V_n + V_n^\vee)V_{n-1}(V_n + V_n^\vee) + tV_{n-1}V_n + tV_{n-1}V_n^\vee. \end{aligned}$$

By Lemmas 3.8, 3.9 and the braid-type relations for V_n and V_n^\vee we obtain from this

$$\begin{aligned} v_{n-1}v_n &= (V_n + V_n^\vee)V_{n-1}(V_n + V_n^\vee)V_{n-1} + tV_nV_{n-1} + tV_n^\vee V_{n-1} \\ &= v_nV_{n-1}v_nV_{n-1} - tv_nV_{n-1} \\ &= v_nv_{n-1}. \end{aligned}$$

Next, we show that $v_nv_i = v_iv_n$ by backward induction on i . The statement is true for $i = n-1$. Suppose that $v_nv_{i+1} = v_{i+1}v_n$ for some $i \in [1, n-2]$, then it follows from (3.5) and Lemma 3.8 that

$$\begin{aligned} v_nv_i &= v_nV_iv_{i+1}V_i - tv_nV_i \\ &= V_iv_nv_{i+1}V_i - tV_iv_n \\ &= V_iv_{i+1}v_nV_i - tV_iv_n \\ &= V_iv_{i+1}V_iv_n - tV_iv_n \\ &= v_iv_n. \end{aligned}$$

This proves the induction step.

Finally, let $i \in [1, n-1]$. A similar induction argument as above shows that $v_iv_j = v_jv_i$ for all $j \in [i+1, n]$. \square

3.2. The duality isomorphism

Recall from (2.2) that we identified the multiplicity function \mathbf{t} with the ordered 5-tuple (t_0, u_0, t, t_n, u_n) . We define an involution σ on the multiplicity function \mathbf{t} by

$$\mathbf{t}^\sigma = (u_n, u_0, t, t_n, t_0), \quad (3.8)$$

i.e., the values of \mathbf{t} on the a_0 -orbit and the a_n^\vee -orbit are interchanged. If an object depends on \mathbf{t} we attach a superscript or subscript σ to denote the same object depending on \mathbf{t}^σ , for example, $\mathcal{H}_\sigma = \mathcal{H}(\mathbf{t}^\sigma)$ and we denote the elements in this algebra by Z^σ ($Z \in \mathcal{H}$).

For ease of notations, we will write

$$U_0 = T_0^\vee, \quad U_n = \Xi_{1,n}^\vee.$$

Note that $U_i^2 = u_i^2$, and U_0, U_1, T_i , $i \in [0, n]$, form a set of generators for the algebra \mathcal{H} .

Proposition 3.11. *The assignments*

$$U_0 \mapsto U_0^\sigma, \quad U_n \mapsto T_0^\sigma, \quad T_0 \mapsto U_n^\sigma, \quad T_i \mapsto T_i^\sigma \quad (i \in [1, n]),$$

extend uniquely to an isomorphism $\sigma : \mathcal{H} \rightarrow \mathcal{H}_\sigma$, respectively an anti-isomorphism $\psi : \mathcal{H} \rightarrow \mathcal{H}_\sigma$.

To prove Proposition 3.11 we only need to check that σ and ψ preserve the defining relations for \mathcal{H} , which is a straightforward exercise. The only relation that is not obvious is the braid-type relation for U_n and T_1 ;

$$U_n T_1 U_n T_1 + t U_n T_1 = T_1 U_n T_1 U_n + t T_1 U_n.$$

We prove a slightly more general result which is useful later on. The relation in the following lemma with $U = \Xi_{1,n}^\vee$ and $j = 2$ is the desired braid-type relation for U_n and T_1 .

Lemma 3.12. *For $j \in [2, n]$ the following braid-type relations hold in \mathcal{H} :*

$$U T_{1,j} U T_{1,j} + t U T_{1,j} = T_{1,j} U T_{1,j} U + t T_{1,j} U, \quad U \in \{T_0, T_0^\vee, \Xi_{1,n}, \Xi_{1,n}^\vee\}.$$

Proof. Let us first prove the identity for $U = \Xi_{1,n}$. The identity for $U = \Xi_{1,n}^\vee$ is proved in exactly the same way. Using $T_n = T_{1,n} \Xi_{1,n} T_{1,n}$ and $T_{1,n} T_{1,j} T_{1,n} = T_{j,n}$ we find

$$\Xi_{1,n} T_{1,j} \Xi_{1,n} T_{1,j} + t \Xi_{1,n} T_{1,j} = T_{1,n} (T_n T_{j,n} T_n T_{j,n} + t T_n T_{j,n}) T_{1,n}.$$

We write out $T_{j,n}$ in terms of T_i 's, then by repeated application of $[T_i, T_j] = 0$ and $T_i^2 = 1$ the expression between brackets becomes

$$\begin{aligned} & T_n (T_j \cdots T_{n-1} \cdots T_j) T_n (T_j \cdots T_{n-1} \cdots T_j) + t T_n (T_j \cdots T_{n-1} \cdots T_j) \\ &= T_j \cdots T_{n-2} (T_n T_{n-1} T_n T_{n-1} + t T_n T_{n-1}) T_{n-2} \cdots T_j. \end{aligned}$$

Now use the braid-type relations between T_n and T_{n-1} and reverse the above steps, then we obtain the desired identity.

For $U = T_0, T_0^\vee$ the desired identities for $j = 2$ are precisely the braid-type relation between U and T_1 . For $j \in [3, n]$ the desired identity follow easily from writing $T_{1,j} = T_{j-1} \cdots T_1 \cdots T_{j-1}$, the braid-type relations between U and T_1 , and repeated application of the relations $[U, T_i] = 0$ and $T_i^2 = 1$ for $i \in [2, j-1]$. \square

3.3. The commutative subalgebra \mathcal{P}_Y

For $i \in [1, n]$ we define elements $Y_i \in H$ by

$$Y_i = \Xi_{i,n} + \Xi_{0,i} + t \sum_{j=i+1}^n T_{i,j}. \quad (3.9)$$

From this definition it is clear that

$$Y_n = \Xi_{n,n} + \Xi_{0,n} = T_n + T_{n-1} \cdots T_1 T_0 T_1 \cdots T_{n-1},$$

and

$$Y_{i+1} = T_i Y_i T_i - t T_i.$$

We denote by $\mathcal{P}_Y \subset H \subset \mathcal{H}$ the subalgebra generated by Y_1, \dots, Y_n . From the definitions of Y_i and the algebra $H = H(\mathfrak{t}_{\mathcal{R}})$ it follows that H is generated as an algebra by $Y_i, T_i, i = 1, \dots, n$. Observe that $\sigma(X_i) = -Y_i^\sigma$, so using the duality isomorphism σ it is easy to find properties for the Y_i 's from the

properties of the X_i 's. Note that we also have $\psi(X_i) = -Y_i^\sigma$, where ψ is the duality anti-isomorphism.

Proposition 3.13. $\mathcal{P}_Y = \mathbb{C}[Y_1, \dots, Y_n]$

Proof. This follows from $\mathcal{P}_Y = \sigma(\mathcal{P}_X^\sigma)$, $\mathcal{P}_X^\sigma = \mathbb{C}[X_1^\sigma, \dots, X_n^\sigma]$, and the fact that σ is an isomorphism. \square

Proposition 3.14. *We have the following relations in H ,*

$$T_i p(Y) - (s_i p)(Y) T_i = d_i(Y; \mathbf{t}^\sigma) [p(Y) - (s_i p)(Y)], \quad i \in [1, n],$$

for $p \in \mathcal{P}$.

Proof. This follows from applying the isomorphism σ_σ to the relations (3.1) in \mathcal{H}_σ . \square

Corollary 3.15. *As a vector space, H is isomorphic to $\mathcal{P}_Y \otimes H_0$ and $H_0 \otimes \mathcal{P}_Y$. Consequently,*

$$\mathcal{H} \simeq \mathcal{P}_Y \otimes H_0 \otimes \mathcal{P}_X \simeq \mathcal{P}_X \otimes H_0 \otimes \mathcal{P}_Y$$

as vector spaces.

Remark 3.16. It should be observed that the elements Y_i are not invertible, which is different from the usual (double) affine Hecke algebra.

3.4. The rational GDAHA for type \tilde{D}_4

We end this section by showing that the rational DAHA $\mathcal{H}(\mathbf{t})$ is isomorphic to the rational generalized double affine Hecke algebra (GDAHA) of rank n attached to an affine Dynkin diagram of type \tilde{D}_4 as defined in [3, Definition 2.2.1]. We recall the definition here. Actually, in [3] a rational GDAHA of rank n is associated to any star-like graph that is not a finite Dynkin diagram.

Let $\nu \in \mathbb{C}$, $\mu = (\mu_{i_0}, \mu_{i_1}, \mu_{i_2}, \mu_{i_3}, \mu_{i_4}) \in \mathbb{C}^5$, and let $\gamma_k = \gamma_k(\mu)$ ($k \in [1, 4]$) be complex numbers such that $\sum_{k=1}^4 \gamma_k = \mu_{i_0}$.

Definition 3.17. The algebra $B = B(\mu, \nu)$ is the complex associative algebra generated by $V_{i,k}$ ($i = 1, \dots, n$, $k = 1, \dots, 4$) and the symmetric group \mathcal{S}_n with the following relations: for any $i, j, k \in [1, n]$ with $i \neq j$, and $l, m \in [1, 4]$,

$$\begin{aligned} s_{ij} V_{i,l} &= V_{j,l} s_{ij}, \\ s_{ij} V_{k,l} &= V_{k,l} s_{ij} \quad \text{if } k \neq i, j, \\ (V_{k,l} - \gamma_l)(V_{k,l} - \gamma_l - \mu_{i_r}) &= 0, \\ V_{i,1} + V_{i,2} + V_{i,3} + V_{i,4} &= \nu \sum_{j \neq i} s_{ij}, \\ [V_{i,l}, V_{j,l}] &= \nu(V_{i,l} - V_{j,l}) s_{ij}, \\ [V_{i,l}, V_{j,m}] &= 0, \quad l \neq m, \end{aligned}$$

where $s_{ij} \in \mathcal{S}_n$ denotes the transposition $i \leftrightarrow j$.

The algebra B essentially only depends on the parameters μ and ν , see also [3, Remark 2.2.2]. We are going to show that B is isomorphic to \mathcal{H} for appropriate μ and ν . For this the following lemma is useful.

Lemma 3.18. *The algebra B is generated by $V_l := V_{1,l}$ ($l \in [1, 4]$) and $s_i := s_{i,i+1}$ ($i \in [1, n-1]$), with relations, for $l \in [1, 4]$, $i \in [1, n-1]$ and $j \in [2, n]$,*

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i V_l &= V_l s_i, \quad i \neq 1, \\ (V_l - \gamma_l)(V_l - \gamma_l - \mu_{i_r}) &= 0, \\ V_1 + V_2 + V_3 + V_4 &= \nu \sum_{k=2}^{n-1} s_{1k}, \\ [V_l, s_{1j} V_l s_{1j}] &= \nu [V_l, s_{1j}], \\ [V_l, s_{1j} V_m s_{1j}] &= 0, \quad m \in [1, 4], \quad m \neq l, \end{aligned}$$

where $s_{1j} = s_1 s_2 \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_2 s_1$.

The proof is a straightforward exercise. We can now show that the rational GDAHA $B(\mu, \nu)$, for suitable parameters μ and ν , is isomorphic to the algebra $\mathcal{H}(\mathbf{t})$, with multiplicity function \mathbf{t} given by (2.2).

Theorem 3.19. *Let $\mu(\mathbf{t}) = (\frac{1}{2} - t_0 - u_0 - t_n - u_n, 2t_0, 2u_0, 2t_n, 2u_n)$. The assignments*

$$\begin{aligned} \varphi(s_i) &= T_i, \quad i \in [1, n-1], \\ \varphi(V_1) &= T_0 + \gamma_1 + t_0, \quad \varphi(V_2) = T_0^\vee + \gamma_2 + u_0, \\ \varphi(V_3) &= \Xi_{1,n} + \gamma_3 + t_n, \quad \varphi(V_4) = \Xi_{1,n}^\vee + \gamma_4 + u_n, \end{aligned}$$

extend uniquely to an algebra isomorphism $\varphi : B(\mu(\mathbf{t}), -t) \rightarrow \mathcal{H}(\mathbf{t})$.

Proof. Note that \mathcal{H} is generated as an algebra by the elements $T_0, T_0^\vee, \Xi_{1,n}, \Xi_{1,n}^\vee$ and T_1, \dots, T_{n-1} , hence φ maps generators to generators and is therefore surjective. It is an almost straightforward computation to check that φ preserves the relations for B from Lemma 3.18. Let us remark that the relations

$$[V_l, s_{1j} V_l s_{1j}] = \nu [V_l, s_{1j}], \quad l \in [1, 4], \quad j \in [2, n], \quad (3.10)$$

are equivalent to the braid-type relations from Lemma 3.12.

We define $\tilde{\varphi} : \mathcal{H}(\mathbf{t}) \rightarrow B(\mu(\mathbf{t}), -t)$ on generators by

$$\begin{aligned} \tilde{\varphi}(T_i) &= s_i, \quad i \in [1, n-1], \\ \tilde{\varphi}(T_0) &= V_1 - \gamma_1 - t_0, \quad \tilde{\varphi}(T_0^\vee) = V_2 - \gamma_2 - u_0, \\ \tilde{\varphi}(T_n) &= s_{1n} V_3 s_{1n} - \gamma_3 - t_n, \quad \tilde{\varphi}(T_n^\vee) = s_{1n} V_1 s_{1n} - \gamma_4 - t_n. \end{aligned}$$

It is now easily checked that $\tilde{\varphi}$ preserves the relations for \mathcal{H} from Theorem 3.7, so that $\tilde{\varphi}$, extended to \mathcal{H} as an algebra homomorphism, is the inverse of φ . \square

4. The polynomial representation

In this section we define and study a representation of the rational double affine Hecke algebra \mathcal{H} , defined in the previous section, in terms of difference-reflection operators.

Recall the algebra $H = H(\mathbf{t}_{\mathcal{R}_r})$ from Definition 2.1, which can be characterized by the relations from Proposition 2.2, see Corollary 2.4. Let $\chi : H \rightarrow \mathbb{C}$ be the trivial representation defined by $\chi(T_i) = \chi_i$ with

$$\chi_i = \begin{cases} t_i, & i = 0, n, \\ 1, & i \in [1, n-1]. \end{cases} \quad (4.1)$$

By Proposition 3.2 we have $\mathcal{H} \simeq H \otimes \mathcal{P}$ as vector spaces, where we identified \mathcal{P}_X with \mathcal{P} . Now we define the representation $\pi : \mathcal{H} \rightarrow \text{End}(\mathcal{P})$ to be the induced representation

$$\pi = \text{Ind}_H^{\mathcal{H}}(\chi).$$

We call π the polynomial representation of \mathcal{H} . From the cross-relations (3.1) we find the action of the generators of \mathcal{H} .

Proposition 4.1. *Let $p \in \mathcal{P}$. The actions of the generators T_i ($i \in [0, n]$) and X_i ($i \in [1, n]$) on \mathcal{P} are given by*

$$\begin{aligned} (\pi(T_i)p)(x) &= c_i(x)(s_i p)(x) - d_i(x)p(x), & i \in [0, n], \\ (\pi(X_i)p)(x) &= x_i p(x), & i \in [1, n], \end{aligned}$$

where $c_i(x) = c_i(x; \mathbf{t})$ are rational functions defined by

$$c_i(x) = d_i(x) + \chi_i, \quad i \in [0, n].$$

Explicitly, the rational functions c_i are given by

$$c_i(x) = \begin{cases} \frac{(t_i + u_i + a_i^\vee(x))(t_i - u_i + a_i^\vee(x))}{a_i(x)}, & i = 0, n, \\ \frac{t + a_i(x)}{a_i(x)}, & i \in [1, n-1]. \end{cases}$$

An easy computation shows that

$$c_i(x) + (s_i c_i)(x) = 2\chi_i, \quad i \in [0, n]. \quad (4.2)$$

We will also use the notation, for $\alpha \in \mathcal{R}_r$,

$$c_\alpha(x; \mathbf{t}) = \begin{cases} \frac{((\mathbf{t}(\alpha) + \mathbf{t}(\alpha^\vee) + \alpha^\vee(x))(\mathbf{t}(\alpha) - \mathbf{t}(\alpha^\vee) + \alpha^\vee(x)))}{\alpha(x)}, & \langle \alpha, \alpha \rangle = 4, \\ \frac{\mathbf{t}(\alpha) + \alpha(x)}{\alpha(x)}, & \langle \alpha, \alpha \rangle = 2. \end{cases}$$

Recalling that the multiplicity function \mathbf{t} is given by (2.2), we see that $c_{a_i} = c_i$ for $i \in [0, n]$, and $w c_\alpha = c_{w\alpha}$ for $w \in W$.

Remark 4.2. (i) Note that for $t = 0$ we have $\pi(T_i) = s_i$, $i \in [1, n-1]$. Moreover, setting $(t_0, u_0, t_n, u_n) = (a, -a, a, -a)$ we find

$$\lim_{a \rightarrow \infty} (\pi(t_i^{-1} T_i) p)(x) = (s_i p)(x), \quad i = 0, n,$$

for $p \in \mathcal{P}$.

(ii) Noumi's [14] representation of the affine Hecke algebra of type \tilde{C}_n on $\mathbb{C}[q^{\pm x_1}, \dots, q^{\pm x_n}]$ is given by

$$V_i = k_{a_i} + k_{a_i}^{-1} \frac{(1 - k_{a_i} k_{a_i/2} q^{a_i(x)/2})(1 + k_{a_i} k_{a_i/2}^{-1} q^{a_i(x)/2})}{1 - q^{a_i(x)}} (s_i - 1), \quad i \in [0, n].$$

Here k is a multiplicity function on \mathcal{R}_{nr} , and $k_{a_i/2} = 1$ if $a_i/2 \notin \mathcal{R}_{nr}$. Setting

$$(k_{a_0}, k_{a_0/2}, k_{a_1}, k_{a_n}, k_{a_n/2}) = (-iq^{t_0}, iq^{u_0}, q^{t/2}, -iq^{t_n}, iq^{u_n}),$$

we find formally for $q \rightarrow 1$ (see also [8, Section 2.4] and [1, Sect. 4])

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{1 - iV_i}{1 - q} &= \pi(T_i), \quad i = 0, n, \\ \lim_{q \rightarrow 1} V_i &= \pi(T_i), \quad i \in [1, n-1]. \end{aligned}$$

So the representation π of \mathcal{H} can formally be obtained as a limit of Noumi's representation of the DAHA associated to \mathcal{R}_{nr} , see [16]. In view of the next proposition the algebra \mathcal{H} may be considered as a degeneration of the DAHA associated to \mathcal{R}_{nr} .

Proposition 4.3. *The representation $\pi : \mathcal{H} \rightarrow \text{End}(\mathcal{P})$ is faithful.*

Proof. Suppose that $\pi(\sum_{w \in W} f_w(X) T_{\bar{w}}) = 0$, with $f_w(X) \in \mathcal{P}_X$. We may write $\pi(T_{\bar{w}}) = \sum_{u \leq w} a_{w,u}(x) u$ with $a_{w,u}(x) \in \mathbb{C}(x)$ and $a_{w,w}(x)$ is nonzero. Let $d(x)$ be the product of the denominators of all $a_{w,u}(x)$, then

$$0 = d(x) \sum_{\substack{u, w \in W \\ u \leq w}} f_w(x) a_{w,u}(x) u.$$

We may consider the expression on the right as an element of the degenerate DAHA $\mathfrak{H}(\mathbf{0}) \simeq \mathcal{P} \otimes W$, with $\mathbf{0}$ the multiplicity function which is equal to zero on all W -orbits in \mathcal{R}_r . Now it follows in the same way as in the proof of Proposition 2.3 that $f_w(x)$ is the polynomial identically equal to zero for all $w \in W$, hence π is a faithful representation of \mathcal{H} . \square

The representation π restricted to the subalgebra H gives a representation of H . We are going to decompose \mathcal{P} , considered as a $\pi(H)$ -module, into irreducible $\pi(H)$ -modules. In view of Proposition 4.3 we identify from here on the algebra \mathcal{H} with the algebra $\pi(\mathcal{H}) \subset \text{End}(\mathcal{P})$.

It will be useful to understand the action of the operators T_i on monomials. For $\nu = \sum_{i=1}^n \nu_i \epsilon_i \in \mathbb{Z}_{\geq 0}^n$ we write $|\nu| = \sum_{i=1}^n \nu_i$.

Lemma 4.4. For $\lambda \in \Lambda$ and $\nu = \phi(\lambda)$, we have

$$\begin{aligned} T_0 x^\nu &= \begin{cases} (t_0 + \langle \lambda, \epsilon_1 \rangle) x^\nu + \sum_{|\mu| < |\nu|} c_{\nu\mu} x^\mu, & \langle \lambda, \epsilon_1 \rangle \geq 0, \\ x^{\phi(s_{\epsilon_1}\lambda)} - (t_0 - \langle \lambda, \epsilon_1 \rangle) x^\nu + \sum_{|\mu| < |\nu|} c_{\nu\mu} x^\mu, & \langle \lambda, \epsilon_1 \rangle < 0, \end{cases} \\ T_i x^\nu &= x^{\phi(s_i\lambda)} + \sum_{|\mu| < |\nu|} c_{\nu\mu} x^\mu, \quad i \in [1, n-1], \\ T_n x^\nu &= \begin{cases} t_n x^\nu, & \langle \lambda, \epsilon_n \rangle \geq 0, \\ -x^{\phi(s_n\lambda)} - t_n x^\nu + \sum_{|\mu| < |\nu|} c_{\nu\mu} x^\mu, & \langle \lambda, \epsilon_n \rangle < 0, \end{cases} \end{aligned}$$

for certain coefficients $c_{\nu\mu} \in \mathbb{C}$.

Proof. For T_0 and T_n this is proved in [8, Proposition 2.5]. For T_i , $i \in [1, n-1]$, we write $T_i = s_i + tD_i$, where

$$(D_i p)(x) = \frac{(s_i p)(x) - p(x)}{a_i(x)}, \quad p \in \mathcal{P}.$$

We use $s_i(x^\nu) = x^{s_i\nu}$. Then from $s_i\nu = \nu - ma_i$, where $m = \langle \nu, a_i \rangle$, we obtain

$$\begin{aligned} D_i x^\nu &= \frac{(s_i x)^\nu - x^\nu}{a_i(x)} = \frac{x^\nu (x^{-ma_i} - 1)}{x_{i+1}(x^{a_i} - 1)} \\ &= \begin{cases} -x^{\nu-\epsilon_{i+1}-a_i} - x^{\nu-\epsilon_{i+1}-2a_i} - \dots - x^{\nu-\epsilon_{i+1}-ma_i}, & m > 0, \\ 0, & m = 0, \\ x^{\nu-\epsilon_{i+1}} + x^{\nu-\epsilon_{i+1}+a_i} + \dots + x^{\nu-\epsilon_{i+1}-(1+m)a_i}, & m < 0. \end{cases} \end{aligned}$$

All these terms are of degree lower than $|\nu|$. □

4.1. Intertwiners

For $i \in [0, n]$ we define the elements $S_i \in \mathcal{H}$, called intertwiners, by

$$\begin{aligned} S_0 &= [U_n, a_0(Y)], \\ S_i &= [T_i, a_i(Y)], \quad i \in [1, n]. \end{aligned} \tag{4.3}$$

We define another action of W on $V_{\mathbb{C}}$ by $s_i \cdot x = s_i x$ for $i \in [1, n]$ and

$$s_0 \cdot x = (-x_1 - 1, x_2, \dots, x_n), \quad x = (x_1, \dots, x_n) \in V_{\mathbb{C}}.$$

We also define a dot-action of W on \mathcal{P} by $(w \cdot p)(x) = p(w^{-1} \cdot x)$. Note in particular that the action of the commutative subalgebra $\tau(\Lambda)$ is given by $(\tau(\epsilon_i) \cdot p)(x) = p(x - \epsilon_i)$.

Lemma 4.5. The intertwiners S_i satisfy the following relations in \mathcal{H} :

(i) The braid relations of type \tilde{C}_n

$$\begin{aligned} S_i S_{i+1} S_i S_{i+1} &= S_{i+1} S_i S_{i+1} S_i, \quad i = 0, n-1, \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1}, \quad i \in [1, n-2], \\ S_i S_j &= S_j S_i, \quad i, j \in [0, n], \quad |i - j| \geq 2. \end{aligned}$$

- (ii) The quadratic relations $S_i^2 = q_i(Y)$ with $q_i(x) = -2a_i(x)^2 c_{a_i}(x) c_{-a_i}(x)$, i.e.,

$$q_i(x) = \begin{cases} 4((u_n + u_0)^2 - (\frac{1}{2} + x_1)^2)((u_n - u_0)^2 - (\frac{1}{2} + x_1)^2), & i = 0, \\ 4(t^2 - a_i(x)^2), & i \in [1, n-1], \\ 4((t_n + t_0)^2 - x_n^2)((t_n - t_0)^2 - x_n^2), & i = n. \end{cases}$$

- (iii) For $p \in \mathcal{P}$,

$$S_i p(Y) = (s_i \cdot p)(Y) S_i, \quad i \in [0, n].$$

Proof. First we apply the duality isomorphism $\sigma : \mathcal{H} \rightarrow \mathcal{H}_\sigma$ and then the representation $\pi_\sigma : \mathcal{H}_\sigma \rightarrow \text{End}(\mathcal{P})$, then by (3.1) we have

$$(\pi_\sigma \circ \sigma)(S_i) = -2a_i(x) c_i(x) s_i.$$

Properties (i) and (ii) follow from this after an easy calculation. Property (iii) follows in the same way, using also $(\pi_\sigma \circ \sigma)(p(Y)) = p(-x)$. \square

For $w \in W$ we define intertwiners S_w by $S_1 = 1$, and

$$S_w = S_{i_1} \cdots S_{i_r}$$

for $w = s_{i_1} \cdots s_{i_r} \in W$ a reduced expression. This is independent of the choice of the reduced expression, since the S_i 's satisfy the braid relations of type \tilde{C}_n . From Proposition 4.5 we now find the following relation in \mathcal{H} .

Corollary 4.6. For $p \in \mathcal{P}$ and $w \in W$ the following relation holds in \mathcal{H} :

$$S_w(w^{-1} \cdot p)(Y) = p(Y) S_w.$$

4.2. Nonsymmetric multivariable Wilson polynomials

For $\lambda \in \Lambda$ let λ^+ be the unique element in $W_0 \lambda \cap \Lambda^+$. There is a unique shortest element (with respect to the length-function l on W) $v_\lambda \in W_0$ such that $v_\lambda \cdot \lambda = \lambda^+$. We define $u_\lambda \in W$ by $u_\lambda = \tau(-\lambda) v_\lambda^{-1}$. We give a few useful properties of u_λ , which are proved in [12, Sect. 2.4].

Lemma 4.7. For $\lambda \in \Lambda$ the element u_λ has the following properties:

- (i) $\tau(-\lambda) = u_\lambda v_\lambda$ and $\tau(-\lambda^+) = v_\lambda u_\lambda$.
- (ii) u_λ is the unique shortest element in W satisfying $u_\lambda \cdot 0 = \lambda$.
- (iii) If $a_i(\lambda) \neq 0$ for $i \in [0, n]$, then $s_i u_\lambda = u_{s_i \cdot \lambda}$.
- (iv) If $a_i(\lambda) < 0$ for $i \in [0, n]$, then $l(u_{s_i \cdot \lambda}) = l(u_\lambda) - 1$.
- (v) If $a_i(\lambda) = 0$ for $i \in [1, n]$, then there exists a $j \in [1, n]$ such that $s_i u_\lambda = u_\lambda s_j$.

We now define polynomials labeled by $\lambda \in \Lambda$ generated by the intertwiners from the constant polynomial $1 \in \mathcal{P}$, see also [13].

Definition 4.8. For $\lambda \in \Lambda$ we define the nonsymmetric multivariable Wilson polynomial $p_\lambda \in \mathcal{P}$ by

$$p_\lambda = S_{u_\lambda} 1.$$

We see that p_λ is a nonzero polynomial for any $\lambda \in \Lambda$. Observe that p_λ depends on the five parameters from the multiplicity function \mathbf{t} , see (2.2). To stress this dependence we sometimes write $p_\lambda = p_\lambda(\cdot; \mathbf{t})$. We will show that the polynomials p_λ are eigenfunctions of the Y -operators.

Remark 4.9. We show later on that the polynomials p_λ are nonsymmetric versions of the multivariable Wilson polynomials as defined by van Diejen [20], which justifies the name. In this paper we will often call p_λ a Wilson polynomial.

Note that the constant function $p_0 = 1 \in \mathcal{P}$ satisfies $T_i p_0 = \chi_i p_0$ for $i \in [0, n]$, see (4.1). Now from the definition of the Y -operators we see that

$$Y_i p_0 = \gamma_{0,i} p_0, \quad \gamma_{0,i} = t_0 + t_n + (n - i)t.$$

For $\lambda \in \Lambda$ we define elements $\gamma_\lambda \in V_{\mathbb{C}}$ as follows:

$$\gamma_\lambda = u_\lambda \cdot \gamma_0, \quad \gamma_0 = \sum_{i=1}^n \gamma_{0,i} \epsilon_i.$$

From the definition of u_λ it follows that γ_λ can also be written as $\lambda + v_\lambda^{-1} \cdot \gamma_0$. From here on we assume that the parameters t_0, t_n, t are such that $\gamma_\lambda \neq \gamma_\mu$ if $\mu \neq \lambda$. We have the following useful lemma.

Lemma 4.10. *Let $p \in \mathcal{P}$ and $\lambda \in \Lambda$, then*

- (i) $s_i \cdot \gamma_\lambda = \gamma_{s_i \cdot \lambda}$ for $i \in [0, n]$.
- (ii) $(s_i p)(-\gamma_\lambda) = p(-\gamma_{s_i \cdot \lambda})$ and $(s_i \cdot p)(\gamma_\lambda) = p(\gamma_{s_i \cdot \lambda})$ for $i \in [0, n]$ with $s_i \cdot \lambda \neq \lambda$.

For the proof see [16, Theorem 5.3].

Theorem 4.11. *For $f \in \mathcal{P}$ and $\lambda \in \Lambda$, we have*

$$f(Y)p_\lambda = f(\gamma_\lambda)p_\lambda.$$

Proof. By Corollary 4.6 and Lemma 4.10 we have

$$\begin{aligned} f(Y)p_\lambda &= f(Y)S_{u_\lambda}1 = S_{u_\lambda}(u_\lambda^{-1} \cdot f)(Y)1 \\ &= (u_\lambda^{-1} \cdot f)(\gamma_0)S_{u_\lambda}1 = f(\gamma_{u_\lambda \cdot 0})p_\lambda = f(\gamma_\lambda)p_\lambda. \end{aligned}$$

□

For $\lambda \in \Lambda$ we define eigenspaces

$$\mathcal{P}_\lambda = \{p \in \mathcal{P} \mid f(Y)p = f(\gamma_\lambda)p \text{ for all } f \in \mathcal{P}\}.$$

Clearly, we have $p_\lambda \in \mathcal{P}_\lambda$, so \mathcal{P}_λ is nonempty and nonzero for every $\lambda \in \Lambda$.

Theorem 4.12. *The set $\{p_\lambda \mid \lambda \in \Lambda\}$ is a basis for \mathcal{P} .*

Proof. We first show that the eigenspaces \mathcal{P}_λ are all 1-dimensional. Let $p \in \mathcal{P}_\lambda$. Observe that for $t = 0$ we have $T_i = s_i$, $i \in [1, n-1]$, which gives us for the Y -operators

$$Y_i = s_i \cdots s_{n-1} T_n s_{n-1} \cdots s_i + s_{i-1} \cdots s_1 T_0 s_1 \cdots s_{i-1}.$$

This is the Y -operator for the rank one version of the algebra \mathcal{H} (see [8]), acting only on the i th variable. For any $m \in \mathbb{Z}_{\geq 0}$ the rank one Y -operator has exactly one (up to a multiplicative constant) polynomial eigenfunction of degree m , which is uniquely determined by the coefficient of x^m . This means that for $t = 0$ we have

$$p(x) = c p_{\phi(\lambda_1)}(x_1) p_{\phi(\lambda_2)}(x_2) \cdots p_{\phi(\lambda_n)}(x_n),$$

where p_m denotes a one-variable nonsymmetric Wilson polynomial of degree m , and c is a nonzero constant. So the coefficient c_λ of $x^{\phi(\lambda)}$ in the expansion $p(x) = \sum_{\mu \in \Lambda} c_\mu x^{\phi(\mu)}$ is nonzero after setting $t = 0$, which implies that c_λ is generically nonzero. Now if $\dim \mathcal{P}_\lambda > 1$, there would be a nonzero polynomial in \mathcal{P}_λ for which the coefficient of $x^{\phi(\lambda)}$ is equal to zero, hence $\dim \mathcal{P}_\lambda = 1$.

Finally, we define for $m \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{P}_{(m)} = \text{span} \left\{ x^{\phi(\lambda)} \mid \lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda, \sum_{i=1}^n |\lambda_i| \leq m \right\},$$

then $\mathcal{P} = \bigcup_{m=0}^{\infty} \mathcal{P}_{(m)}$. Since $\sum_i \lambda_i^+ = \sum_i |\lambda_i|$ for all $\lambda \in \Lambda$, we see from Lemma 4.4 that the intertwiners S_i , $i \in [1, n]$, satisfy $S_i \mathcal{P}_{(m)} \subset \mathcal{P}_{(m)}$, while $S_0 \mathcal{P}_{(m)} \subset \mathcal{P}_{(m+1)}$. It follows that $p_\lambda \in \mathcal{P}_{(N)}$ where N is the number of times s_0 occurs in a reduced expression for u_λ . By Lemma 4.7 we have $u_\lambda = v_\lambda^{-1} \tau(-\lambda^+)$ with $v_\lambda \in W_0$, so N is also the number of times s_0 occurs in $\tau(-\lambda^+)$, i.e., $N = \sum_i \lambda_i^+$. So for any $m \in \mathbb{Z}_{\geq 0}$ the set $\{p_\lambda \mid \sum_i |\lambda_i| \leq m\} \subset \mathcal{P}_{(m)}$ has the same cardinality as the basis $\{x^{\phi(\lambda)} \mid \sum_i |\lambda_i| \leq m\}$ of $\mathcal{P}_{(m)}$. Since the polynomials p_λ , $\sum_i |\lambda_i| \leq m$, are eigenfunctions of the Y -operators for pairwise different eigenvalues, they are linear independent in $\mathcal{P}_{(m)}$, hence the set $\{p_\lambda \mid \sum_i |\lambda_i| \leq m\}$ forms a basis for $\mathcal{P}_{(m)}$. \square

We have the following corollary of the proof of Theorem 4.12.

Corollary 4.13. *The eigenspaces \mathcal{P}_λ , $\lambda \in \Lambda$, are all one-dimensional. Furthermore, a polynomial in \mathcal{P}_λ is uniquely determined by the coefficient of $x^{\phi(\lambda)}$.*

We define for $\lambda \in \Lambda^+$,

$$\mathcal{P}(\lambda) = \text{span}\{p_\mu \mid \mu \in W_0 \lambda\}.$$

We are going to show that $\mathcal{P}(\lambda)$ is an irreducible $\pi(H)$ -module.

Lemma 4.14. *Let $\lambda \in \Lambda$ and $i \in [0, n]$, then*

$$S_i p_\lambda = \begin{cases} 0, & a_i(\lambda) = 0, \\ p_{s_i \cdot \lambda}, & a_i(\lambda) > 0, \\ q_i(\gamma_{s_i \cdot \lambda}) p_{s_i \cdot \lambda}, & a_i(\lambda) < 0. \end{cases}$$

Proof. Let us first assume that $a_i(\lambda) \neq 0$, i.e., $s_i \cdot \lambda \neq \lambda$. We have $S_i p_\lambda = S_i S_{u_\lambda} 1$. Let $s_{i_1} \cdots s_{i_r}$ be a reduced expression for u_λ . We need to find out if $s_i s_{i_1} \cdots s_{i_r}$ is a reduced expression for $s_i u_\lambda$. By Lemma 4.7 we have $u_{s_i \cdot \lambda} = s_i u_\lambda$. If $l(u_{s_i \cdot \lambda}) > l(u_\lambda)$, then $s_i s_{i_1} \cdots s_{i_r}$ is a reduced expression for $u_{s_i \cdot \lambda}$. In this case $S_i S_{u_\lambda} = S_{u_{s_i \cdot \lambda}}$, which implies $S_i p_\lambda = p_{s_i \cdot \lambda}$. If $l(u_{s_i \cdot \lambda}) < l(u_\lambda)$, then there exists a reduced expression for u_λ which starts with s_i . This shows that $S_i S_{u_\lambda} = S_i^2 S_{u_{s_i \cdot \lambda}}$. From Lemma 4.5 and Theorem 4.11 we then obtain $S_i p_\lambda = S_i^2 p_{s_i \cdot \lambda} = q_i(\gamma_{s_i \cdot \lambda}) p_{s_i \cdot \lambda}$. By Lemma 4.7 $a_i(\lambda) < 0$ implies $l(u_{s_i \cdot \lambda}) < l(u_\lambda)$. Replacing λ by $s_i \cdot \lambda$ then shows that $a_i(\lambda) > 0$ implies $l(u_{s_i \cdot \lambda}) > l(u_\lambda)$. This proves the lemma in case $a_i(\lambda) \neq 0$.

Now suppose that $a_i(\lambda) = 0$, i.e., $s_i \cdot \lambda = \lambda$, which only happens if $i \neq 0$. In this case we have $(s_i u_\lambda) \cdot 0 = s_i \cdot \lambda = \lambda$. Since u_λ is the shortest element in W satisfying $u_\lambda \cdot 0 = \lambda$, we have $l(s_i u_\lambda) > l(u_\lambda)$. Furthermore, by Lemma 4.7 there exists a $j \in [1, n]$ such that $s_i u_\lambda = u_\lambda s_j$. This shows that $S_i p_\lambda = S_i S_{u_\lambda} 1 = S_{u_\lambda} S_j 1$. Observe that $S_j 1 = [T_j, a_j(Y)] 1 = 0$, since $T_j 1 = \chi_j$, hence $S_i p_\lambda = 0$. \square

The following result assures us that $T_i \mathcal{P}(\lambda) \subset \mathcal{P}(\lambda)$ for $i \in [1, n]$. Clearly, we also have $Y_i \mathcal{P}(\lambda) \subset \mathcal{P}(\lambda)$ for $i \in [1, n]$. Since the Y -operators are diagonalized by p_μ , a nonempty invariant subspace $\tilde{\mathcal{P}} \subset \mathcal{P}(\lambda)$ contains an element p_μ for some $\mu \in W_0 \lambda$. Repeated application of the intertwiners S_i , $i \in [1, n]$, then shows that for every $\nu \in W_0 \lambda$ we have $p_\nu \in \tilde{\mathcal{P}}$, hence $\mathcal{P}(\lambda)$ is irreducible.

Lemma 4.15. *Let $\lambda \in \Lambda$ and $i \in [1, n]$. If $a_i(\lambda) \neq 0$, then*

$$T_i p_\lambda = A_{i,\lambda} p_\lambda + B_{i,\lambda} p_{s_i \cdot \lambda},$$

with $A_{i,\lambda} = d_i^\sigma(\gamma_\lambda)$ and

$$B_{i,\lambda} = \begin{cases} \frac{1}{2a_i(\gamma_\lambda)}, & a_i(\lambda) > 0, \\ \frac{q_i(\gamma_{s_i \cdot \lambda})}{2a_i(\gamma_\lambda)}, & a_i(\lambda) < 0. \end{cases}$$

If $a_i(\lambda) = 0$, then $T_i p_\lambda = \chi_i p_\lambda$.

Proof. Let $i \in [1, n]$ and $\lambda \in \Lambda$. By Theorem 4.12 $T_i p_\lambda$ can be expanded in terms of p_μ , $\mu \in \Lambda$. From the cross-relation in Proposition 3.14 we find for any $f \in \mathcal{P}$,

$$(f(Y) - (s_i f)(\gamma_\lambda)) T_i p_\lambda = (f(\gamma_\lambda) - (s_i f)(\gamma_\lambda)) d_i^\sigma(\gamma_\lambda) p_\lambda. \quad (4.4)$$

Suppose that $a_i(\lambda) \neq 0$, then $s_i \cdot \lambda \neq \lambda$. In this case we find from Theorem 4.11 and Lemma 4.10 that $T_i p_\lambda$ is indeed of the form $A_{i,\lambda} p_\lambda + B_{i,\lambda} p_{s_i \cdot \lambda}$ with $A_{i,\lambda} = d_i^\sigma(\gamma_\lambda)$. In order to find $B_{i,\lambda}$ we use the intertwiner S_i . By Lemma 4.14 we have $S_i p_\lambda = g_{i,\lambda} p_{s_i \cdot \lambda}$ where

$$g_{i,\lambda} = \begin{cases} 1, & a_i(\lambda) > 0, \\ q_i(\gamma_{s_i \cdot \lambda}), & a_i(\lambda) < 0. \end{cases}$$

On the other hand, using $S_i = [T_i, a_i(Y)]$ we have

$$S_i p_\lambda = (a_i(\gamma_\lambda) - a_i(\gamma_{s_i \cdot \lambda})) B_{i,\lambda} p_{s_i \cdot \lambda},$$

hence $B_{i,\lambda} = g_{i,\lambda}/2a_i(\gamma_\lambda)$.

Now suppose that $a_i(\lambda) = 0$, then Theorem 4.11 and (4.4) imply that $T_i p_\lambda = d_i^\sigma(\gamma_\lambda) p_\lambda$. Since $s_i \cdot \lambda = \lambda$ it follows from the definition of γ_λ that $a_i(\gamma_\lambda) = t$ for $i \in [1, n-1]$ and $a_n(\gamma_\lambda) = 2t_0 + 2t_n$. From the explicit expression for $d_i^\sigma(x)$ we then find $d_i^\sigma(\gamma_\lambda) = \chi_i$. \square

From Lemma 4.15 and Theorem 4.11 we obtain the following result.

Proposition 4.16. *The center of H is equal to $\mathcal{P}_Y^{W_0}$.*

Proof. From Proposition 3.14 it follows directly that the subalgebra $\mathcal{P}_Y^{W_0}$ commutes with generators T_i of H , hence $\mathcal{P}_Y^{W_0}$ is contained in the center of the algebra H .

Let Z be an element in the center of H . By Corollary 3.15 we may write $Z = \sum_{w \in W_0} T_{\bar{w}} f_w(Y)$ with $f_w(Y) \in \mathcal{P}_Y$. Since Z commutes with \mathcal{P}_Y , Z acts as a constant on p_λ , hence

$$\sum_{w \in W_0} f_w(\gamma_\lambda) T_{\bar{w}} p_\lambda = c p_\lambda,$$

for some nonzero constant c . By Lemma 4.15 we have $T_{\bar{w}} p_\lambda = \sum_{v \leq w} c_{wv}^\lambda p_{v \cdot \lambda}$ for certain coefficients $c_{wv}^\lambda \in \mathbb{C}$, and $c_{ww}^\lambda \neq 0$ for infinitely many $\lambda \in \Lambda$. This gives

$$\sum_{\substack{v, w \in W_0 \\ v \leq w}} f_w(\gamma_\lambda) c_{wv}^\lambda p_{v \cdot \lambda} = c p_\lambda,$$

which implies that for $v \neq 1$

$$\sum_{\substack{w \in W_0 \\ w \geq v}} f_w(\gamma_\lambda) c_{wv}^\lambda = 0.$$

Let $u \neq 1$ be a maximal element such that $f_u \neq 0$, then $c_{uu}^\lambda = 0$ for infinitely many $\lambda \in \Lambda$. We conclude that only f_1 is nonzero. Being a central element, $Z = f_1(Y)$ commutes with every intertwiner S_i , $i \in [1, n]$. Lemma 4.5 then implies that $s_i f_1 = f_1$, hence $f_1(Y) \in \mathcal{P}_Y^{W_0}$. \square

Theorem 4.17. *The decomposition*

$$\mathcal{P} = \bigoplus_{\lambda \in \Lambda^+} \mathcal{P}(\lambda) \tag{4.5}$$

is the multiplicity-free, irreducible decomposition of \mathcal{P} as a $\pi(H)$ -module. Moreover, (4.5) gives the decomposition of \mathcal{P} into isotypical components under the action of the center $\mathcal{P}_Y^{W_0}$ of H , and the central character is given by $\chi_\lambda(p(Y)) = p(\gamma_\lambda)$ for $p(Y) \in \mathcal{P}_Y^{W_0}$.

5. Nonsymmetric multivariable Wilson polynomials

In this section we derive a few important properties for the nonsymmetric multivariable Wilson polynomials, such as orthogonality relations, evaluation formulas and the duality property. These properties are obtained in the same way as in the case of the Koornwinder polynomials, see [14, 16–18].

5.1. Duality

We write $x_\lambda = \gamma_\lambda(\mathbf{t}^\sigma)$, $\lambda \in \Lambda$, for the spectrum of the operators $Y^\sigma \in \mathcal{H}_\sigma$. Here the dual multiplicity function \mathbf{t}^σ is defined by (3.8). We define evaluation mappings $\text{Ev} : \mathcal{H} \rightarrow \mathbb{C}$ and $\widetilde{\text{Ev}} : \mathcal{H}_\sigma \rightarrow \mathbb{C}$ by

$$\begin{aligned}\text{Ev}(Z) &= (Z(1))(-x_0), \quad Z \in \mathcal{H}, \\ \widetilde{\text{Ev}}(\tilde{Z}) &= (\tilde{Z}(1))(-\gamma_0), \quad \tilde{Z} \in \mathcal{H}_\sigma.\end{aligned}$$

The two evaluation mappings are related via the duality anti-isomorphism $\psi : \mathcal{H} \rightarrow \mathcal{H}_\sigma$ from Proposition 3.11 by

$$\text{Ev}(Z) = \widetilde{\text{Ev}}(\psi(Z)), \quad Z \in \mathcal{H}. \quad (5.1)$$

Indeed, for $Z = f(X)T_w g(Y)$ with $f, g \in \mathcal{P}$ and $w = s_{i_1} \cdots s_{i_r} \in W_0$ a reduced expression, we have, using $\psi(X_i) = -Y_i^\sigma$ and $\psi(T_i) = T_i^\sigma$ for $i \in [1, n]$,

$$\begin{aligned}\widetilde{\text{Ev}}(\psi(Z)) &= (g(-X^\sigma)T_{w^{-1}}^\sigma f(-Y^\sigma)(1))(-\gamma_0) \\ &= \chi_\sigma(T_{w^{-1}}^\sigma) f(-x_0) g(\gamma_0) \\ &= (f(X)T_w g(Y)(1))(-x_0) \\ &= \text{Ev}(Z).\end{aligned}$$

Here we use the reduced expression $w^{-1} = s_{i_r} \cdots s_{i_1}$, and χ is the trivial representation of H_0 . Note that $\chi(T_w) = \chi_\sigma(T_{w^{-1}})$ for any word $w \in W_0$ (and $\chi_\sigma : H_0^\sigma \rightarrow \mathbb{C}$ is defined in the same way as $\chi : H_0 \rightarrow \mathbb{C}$). By the PBW-property for \mathcal{H} from Corollary 3.15 we then see that (5.1) holds for all $Z \in \mathcal{H}$.

With the evaluation mappings and the duality anti-isomorphism $\psi : \mathcal{H} \rightarrow \mathcal{H}_\sigma$ we construct two pairings $B : \mathcal{H} \times \mathcal{H}_\sigma \rightarrow \mathbb{C}$ and $\tilde{B} : \mathcal{H}_\sigma \times \mathcal{H} \rightarrow \mathbb{C}$ by

$$B(Z, \tilde{Z}) = \text{Ev}(\psi_\sigma(\tilde{Z})Z), \quad \tilde{B}(\tilde{Z}, Z) = \widetilde{\text{Ev}}(\psi(Z)\tilde{Z}),$$

where $Z \in \mathcal{H}$ and $\tilde{Z} \in \mathcal{H}_\sigma$. We have the following properties for these pairings.

Proposition 5.1. *Let $Z, Z_1, Z_2 \in \mathcal{H}$, $\tilde{Z}, \tilde{Z}_1, \tilde{Z}_2 \in \mathcal{H}_\sigma$ and $p \in \mathcal{P}$. Then*

- (i) $B(Z, \tilde{Z}) = \tilde{B}(\tilde{Z}, Z)$;
- (ii) $B(Z_1 Z_2, \tilde{Z}) = B(Z_2, \psi(Z_1)\tilde{Z})$ and $B(Z, \tilde{Z}_1 \tilde{Z}_2) = B(\psi_\sigma(\tilde{Z}_1)Z, \tilde{Z}_2)$;
- (iii) $B((Zp)(X), \tilde{Z}) = B(Zp(X), \tilde{Z})$ and $B(Z, (\tilde{Z}p)(X^\sigma)) = B(Z, \tilde{Z}p(X^\sigma))$.

Here $p(X)$ is the multiplication operator $(p(X)f)(x) = p(x)f(x)$ for $f \in \mathcal{P}$.

Proof. Properties (i) and (ii) follow from (5.1). Property (iii) is an immediate consequence of the identity $((Zp)(X))(1) = Zp = Z(p(X)(1))$ in \mathcal{P} . \square

The evaluation $\text{Ev}(p_\lambda(X)) = p_\lambda(-x_0)$ of the nonsymmetric Wilson polynomial is nonzero for generic parameters $\mathbf{t} = (t_0, u_0, t, t_n, u_n)$. Indeed, for $t = 0$, the polynomial p_λ becomes a product of one-variable nonsymmetric Wilson polynomials $p_{\phi(\lambda_i)}$ which are all nonzero at $-x_{0,i} = -(t_n + u_n)$, see [8, Proposition 3.14]. In the next subsection we determine an explicit expression for $p_\lambda(-x_0)$.

Definition 5.2. For $\lambda \in \Lambda$ we write $E(x, \gamma_\lambda; \mathbf{t}) = E(x, \gamma_\lambda)$ for the constant multiple of the nonsymmetric Wilson polynomial $p_\lambda(x; \mathbf{t})$ that takes the value 1 at $x = -x_0$.

As a consequence of Proposition 5.1 we obtain

$$B(p(X^\sigma), E(X, \gamma_\lambda)) = B(1, (p(-Y)E(\cdot, \gamma_\lambda))(X)) = p(-\gamma_\lambda)B(1, E(X, \gamma_\lambda)),$$

for any $p \in \mathcal{P}$ and $\lambda \in \Lambda$. Note that $B(1, E(X, \gamma_\lambda)) = \text{Ev}(E(X, \gamma_\lambda)) = 1$, so we have

$$f(-\gamma_\lambda) = B(f(X^\sigma), E(X, \gamma_\lambda)), \quad g(-x_\mu) = \tilde{B}(g(X), E_\sigma(X^\sigma, x_\mu)), \quad (5.2)$$

for $f, g \in \mathcal{P}$ and $\lambda, \mu \in \Lambda$. The second identity is derived in the same way as the first. This immediately leads to the duality property for the renormalized nonsymmetric Wilson polynomials.

Theorem 5.3. For $\lambda, \mu \in \Lambda$ the renormalized nonsymmetric Wilson polynomials satisfy the duality property

$$E(-x_\mu, \gamma_\lambda; \mathbf{t}) = E(-\gamma_\lambda, x_\mu; \mathbf{t}^\sigma).$$

Proof. This follows from Proposition 5.1(i) if we set $f = E_\sigma(\cdot, x_\mu)$ and $g = E(\cdot, \gamma_\lambda)$ in (5.2). \square

As a consequence of the duality property we obtain that the actions of T_i , $i \in [1, n]$, and U_n on the renormalized nonsymmetric Wilson polynomials can be written as difference-reflection operators acting on the spectral parameters.

Proposition 5.4. Let $\lambda \in \Lambda$, then

$$\begin{aligned} (U_n E(\cdot, \gamma_\lambda))(x) &= c_0^\sigma(-\gamma_\lambda)E(x, \gamma_{s_0 \cdot \lambda}) - d_0^\sigma(-\gamma_\lambda)E(x, \gamma_\lambda), \\ (T_i E(\cdot, \gamma_\lambda))(x) &= c_i^\sigma(-\gamma_\lambda)E(x, \gamma_{s_i \cdot \lambda}) - d_i^\sigma(-\gamma_\lambda)E(x, \gamma_\lambda), \quad i \in [1, n]. \end{aligned}$$

Proof. It is enough to prove the result for $x = -x_\mu$ for all $\mu \in \Lambda$. By (5.2) and Proposition 5.1 we have for $\lambda, \mu \in \Lambda$

$$\tilde{B}(E(X, \gamma_\lambda), (T_i^\sigma E_\sigma(\cdot, x_\mu))(X^\sigma)) = \begin{cases} (U_n E(\cdot, \gamma_\lambda))(-x_\mu), & i = 0, \\ (T_i E(\cdot, \gamma_\lambda))(-x_\mu), & i \in [1, n]. \end{cases}$$

Writing out $T_i^\sigma E_\sigma(\cdot, x_\mu)$ we find using Proposition 5.1

$$\begin{aligned} & \tilde{B}(E(X, \gamma_\lambda), (T_i^\sigma E_\sigma(\cdot, x_\mu))(X^\sigma)) \\ &= \tilde{B}(E(X, \gamma_\lambda), c_i^\sigma(X^\sigma)(s_i E_\sigma(\cdot, x_\mu))(X^\sigma) - d_i^\sigma(X^\sigma)E_\sigma(X^\sigma, x_\mu)) \\ &= B(c_i^\sigma(X^\sigma)(s_i E_\sigma(\cdot, x_\mu))(X) - d_i^\sigma(X^\sigma)E_\sigma(X^\sigma, x_\mu), E(X, \gamma_\lambda)) \\ &= c_i^\sigma(-\gamma_\lambda)(s_i E_\sigma(\cdot, x_\mu))(-\gamma_\lambda) - d_i^\sigma(-\gamma_\lambda)E_\sigma(-\gamma_\lambda, x_\mu) \\ &= c_i^\sigma(-\gamma_\lambda)E(-x_\mu, \gamma_{s_i \cdot \lambda}) - d_i^\sigma(-\gamma_\lambda)E(-x_\mu, \gamma_\lambda). \end{aligned}$$

The last two lines follow from (5.2), Lemma 4.10 and the duality property for the nonsymmetric Wilson polynomials, Theorem 5.3. \square

If $a_i(\lambda) \neq 0$ we already know from Lemma 4.14 that $S_i E(\cdot, \gamma_\lambda) = c E(\cdot, \gamma_{s_i \cdot \lambda})$ for some nonzero constant c . Using Proposition 5.4 the constant c can be calculated.

Corollary 5.5. *Let $i \in [0, n]$, and let $\lambda \in \Lambda$ such that $s_i \cdot \lambda \neq \lambda$, then*

$$S_i E(\cdot, \gamma_\lambda) = 2a_i(\gamma_\lambda)c_i^\sigma(-\gamma_\lambda)E(\cdot, \gamma_{s_i \cdot \lambda}).$$

5.2. The evaluation formula

Using Corollary 5.5 we can determine the value of $p_\lambda(-x_0)$.

Theorem 5.6. *For $\lambda \in \Lambda$,*

$$p_\lambda(-x_0) = \prod_{\alpha \in \mathcal{R}_r^+ \cap u_\lambda^{-1} \mathcal{R}_r^-} K_\alpha(-\gamma_0),$$

where

$$K_\alpha(x) = 2\alpha(-x)c_\alpha^\sigma(x).$$

Proof. By the definition of $E(x, \gamma_\lambda)$ we have $p_\lambda = p_\lambda(-x_0)E(\cdot, \gamma_\lambda)$. Recall that $p_\lambda = S_{u_\lambda} 1$, and let $u_\lambda = s_{i_1} \cdots s_{i_r}$ be a reduced expression. Using Corollary 5.5 we now find

$$p_\lambda(-x_0) = \prod_{k=1}^r K_{a_{i_k}} \left(-\gamma_{(s_{i_{k+1}} \cdots s_{i_r}) \cdot 0} \right).$$

Recall from Lemma 4.7 that u_λ is the unique shortest element in W such that $u_\lambda \cdot 0 = \lambda$, so $(s_{i_k} s_{i_{k+1}} \cdots s_{i_r}) \cdot 0 \neq (s_{i_{k+1}} \cdots s_{i_r}) \cdot 0$. By Lemma 4.10 we have $K_\alpha(-\gamma_{w \cdot 0}) = K_{w^{-1}\alpha}(-\gamma_0)$ for $w \in W$, so we can write $p_\lambda(-x_0)$ as a product over the set $S = \{\beta_1, \dots, \beta_r\}$ with $\beta_k = s_{i_r} \cdots s_{i_{k+1}} a_{i_k}$ for $k \in [1, r]$. It is well known that the set S is equal to $\mathcal{R}_r^+ \cap u_\lambda^{-1} \mathcal{R}_r^-$, which proves the result. \square

5.3. Orthogonality relations

With the multiplicity function \mathbf{t} given by (2.2) we associate the Wilson parameters a, b, c, d given by

$$(a, b, c, d) = \left(t_n + u_n, t_n - u_n, t_0 + u_0 + \frac{1}{2}, t_0 - u_0 + \frac{1}{2} \right).$$

We assume from here on that $a, b, c, d, t > 0$. We define $\Delta(\cdot) = \Delta(\cdot; \mathbf{t})$ and $\Delta_+(\cdot) = \Delta_+(\cdot; \mathbf{t})$ by

$$\Delta_+(x; \mathbf{t}) = \prod_{1 \leq j < k \leq n} \frac{\Gamma(t \pm x_j \pm x_k)}{\Gamma(\pm x_j \pm x_k)} \prod_{j=1}^n \frac{\Gamma(a \pm x_j) \Gamma(b \pm x_j) \Gamma(c \pm x_j) \Gamma(d \pm x_j)}{\Gamma(\pm 2x_j)},$$

and

$$\Delta(x; \mathbf{t}) = c_+(x; \mathbf{t}) \Delta_+(x; \mathbf{t}),$$

where

$$\begin{aligned} c_+(x; \mathbf{t}) &= \prod_{\alpha \in \Sigma^-} c_\alpha(x; \mathbf{t}) \\ &= \prod_{1 \leq j < k \leq n} \frac{(t - x_j + x_k)(t - x_j - x_k)}{(x_k + x_j)(x_k - x_j)} \prod_{j=1}^n \frac{(a - x_j)(b - x_j)}{-2x_j} \end{aligned} \quad (5.3)$$

To the weight functions Δ and Δ_+ we associate two nondegenerate bilinear forms on \mathcal{P} by

$$\begin{aligned} \langle f, g \rangle_{\mathbf{t}} &= \int_{(\mathbb{i}\mathbb{R})^n} f(x) g(x) \Delta(x; \mathbf{t}) dx, \\ \langle f, g \rangle_{\mathbf{t}}^+ &= \int_{(\mathbb{i}\mathbb{R})^n} f(x) g(x) \Delta_+(x; \mathbf{t}) dx, \end{aligned}$$

where $dx = (2\pi i)^{-n} dx_1 dx_2 \cdots dx_n$, and $\mathbb{i}\mathbb{R}$ has the standard orientation. The constant terms can be given explicitly;

$$\langle 1, 1 \rangle_{\mathbf{t}}^+ = 2^n n! \prod_{j=1}^n \frac{\Gamma(tj) \prod_{1 \leq k < l \leq 4} \Gamma(v_k + v_l + (j-1)t)}{\Gamma(t) \Gamma(v_1 + v_2 + v_3 + v_4 + (n+j-2)t)},$$

where $(v_1, v_2, v_3, v_4) = (a, b, c, d)$, see [9]. From Stirling's formula it follows that the polynomials are integrable on $(\mathbb{i}\mathbb{R})^n$ with respect to both Δ and Δ_+ . Let us remark that the weight Δ_+ is positive on $(\mathbb{i}\mathbb{R})^n$ under the current conditions on \mathbf{t} .

Let $\iota : \mathcal{H} \rightarrow \mathcal{H}$ be the anti-isomorphism defined by $\iota(T_i) = T_i$, $i \in [0, n]$, and $\iota(X_j) = X_j$, $j \in [1, n]$.

Lemma 5.7. *Let $f, g \in \mathcal{P}$ and $Z \in \mathcal{H}$, then $\langle Zf, g \rangle_{\mathbf{t}} = \langle f, \iota(Z)g \rangle_{\mathbf{t}}$.*

Proof. It suffices to show that the generators T_i and X_j are symmetric with respect to $\langle \cdot, \cdot \rangle_{\mathbf{t}}$. For the X_j 's this is obvious, so we only need to verify it for the T_i 's. Writing out $\langle T_i f, g \rangle_{\mathbf{t}}$ we see that the proof of the lemma boils down to proving the following identity:

$$\int_{(\mathbb{i}\mathbb{R})^n} (s_i f)(x) g(x) c_i(x) \Delta(x) dx = \int_{(\mathbb{i}\mathbb{R})^n} f(x) (s_i g)(x) c_i(x) \Delta(x) dx. \quad (5.4)$$

Using the definitions of c_i , $0 = 1, \dots, n$, and Δ one checks that the function $c_i(x)\Delta(x)$ is invariant under the action of s_i . Now for $i = 1, \dots, n$ the required identity (5.4) follows from substituting $x \mapsto s_i x$ on the left hand side. For $i = 0$ we denote $y = (x_2, \dots, x_n)$ and $\alpha(x) = \alpha(x_1, y)$ for any function α depending on x_1, \dots, x_n . We substitute $x_1 \mapsto 1 - y_1$ in the left hand side of (5.4), then this becomes

$$\int_{y_1 \in 1+i\mathbb{R}} \int_{y \in (i\mathbb{R})^{n-1}} f(y_1, y) g(1 - y_1, y) c_0(y_1, y) \Delta(y_1, y) dy dy_1.$$

The function $y_1 \mapsto c_0(y_1, y) \Delta(y_1, y)$ has no poles inside the strip $\{0 \leq \Re(y_1) \leq 1\}$ due to the conditions on the parameters. Furthermore, using Stirling's formula it follows that the integral over the line segment $\{y_1 = x + iB \mid 0 \leq x \leq 1\}$ vanishes for $B \rightarrow \pm\infty$. By Cauchy's theorem we may then shift the contour $1 + i\mathbb{R}$ to $i\mathbb{R}$ without changing the outcome of the integral. This proves identity (5.4) for $i = 0$. \square

The Y -operators satisfy $\iota(Y_i) = Y_i$, $i \in [1, n]$. From the previous lemma we then immediately obtain orthogonality relations for the Wilson polynomials. Combining this with Theorem 4.12 we have the following result.

Theorem 5.8. *The nonsymmetric Wilson polynomials form an orthogonal basis for \mathcal{P} with respect to $\langle \cdot, \cdot \rangle_{\mathbf{t}}$, i.e.,*

$$\langle p_\lambda, p_\mu \rangle_{\mathbf{t}} = 0, \quad \lambda \neq \mu.$$

Remark 5.9. There is a fundamental difference with the DAHA of type (C^\vee, C) . In that algebra the identity $\iota(Y_i) = Y_i$ is not valid for the corresponding Y -operators. Therefore, the polynomial eigenfunctions of the Y -operators, the nonsymmetric Koornwinder polynomials, satisfy biorthogonality relations, see [16].

Next we compute the diagonal terms $\langle E(\cdot, \gamma_\lambda), E(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}}$.

Lemma 5.10. *For $\lambda \in \Lambda$ and $i \in [0, n]$,*

$$\langle E(\cdot, \gamma_\lambda), E(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}} = \frac{c_{-a_i}^\sigma(\gamma_\lambda)}{c_{a_i}^\sigma(\gamma_\lambda)} \langle E(\cdot, \gamma_{s_i \cdot \lambda}), E(\cdot, \gamma_{s_i \cdot \lambda}) \rangle_{\mathbf{t}}.$$

Proof. By Lemma 5.7 and the definition (4.3) of the intertwiners S_i we have

$$\langle S_i f, g \rangle_{\mathbf{t}} = -\langle f, S_i g \rangle_{\mathbf{t}}, \quad f, g \in \mathcal{P}.$$

Since $S_i E(\cdot, \gamma_\lambda) = b_{\lambda, i} E(\cdot, \gamma_{s_i \cdot \lambda})$, with $b_{\lambda, i}$ given in Corollary 5.5, we have

$$\langle S_i E(\cdot, \gamma_\lambda), S_i E(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}} = b_{\lambda, i}^2 \langle E(\cdot, \gamma_{s_i \cdot \lambda}), E(\cdot, \gamma_{s_i \cdot \lambda}) \rangle_{\mathbf{t}}.$$

On the other hand, by Lemma 4.5 we have

$$\begin{aligned} \langle S_i E(\cdot, \gamma_\lambda), S_i E(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}} &= -\langle E(\cdot, \gamma_\lambda), S_i^2 E(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}} \\ &= -q_i(\gamma_\lambda) \langle E(\cdot, \gamma_\lambda), E(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}}. \end{aligned}$$

Now the result follows from $-b_{\lambda, i}^2/q_i(\gamma_\lambda) = c_{-a_i}^\sigma(\gamma_\lambda)/c_{a_i}^\sigma(\gamma_\lambda)$. \square

For $\lambda \in \Lambda$ we define

$$N(\gamma_\lambda; \mathbf{t}) = \frac{1}{\langle 1, 1 \rangle_{\mathbf{t}}} \prod_{\alpha \in \mathcal{R}_r^+ \cap u_\lambda \mathcal{R}_r^-} \frac{c_{-\alpha}^\sigma(\gamma_\lambda)}{c_\alpha^\sigma(\gamma_\lambda)}$$

From $E(\cdot, \gamma_0) = 1$, Lemma 5.10 and arguments similar as in the proof of Theorem 5.6 we see that $N(\gamma_\lambda)^{-1}$ is the ‘quadratic norm’ for $E(\cdot, \gamma_\lambda)$.

Theorem 5.11. *For $\lambda \in \Lambda$,*

$$\langle E(\cdot, \gamma_\lambda), E(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}} = \frac{1}{N(\gamma_\lambda)}.$$

5.4. The Fourier transform

We define

$$\text{Spec}(-Y) = \{-\gamma_\lambda \mid \lambda \in \Lambda\}.$$

Let F be the vector space of complex-valued functions on $\text{Spec}(-Y)$ with finite support. A linear basis for F is formed by the functions δ_λ , $\lambda \in \Lambda$, given by

$$\delta_\lambda(-\gamma_\mu) = \begin{cases} 0, & \mu \neq \lambda, \\ 1, & \mu = \lambda. \end{cases}$$

We define a bilinear form $[\cdot, \cdot] : F \times F \rightarrow \mathbb{C}$ by

$$[f, g]_{\mathbf{t}} = \sum_{\gamma \in \text{Spec}(-Y)} f(\gamma)g(\gamma)N(-\gamma; \mathbf{t})$$

The functions δ_λ , $\lambda \in \Lambda$, are orthogonal with respect to this bilinear form.

We define the polynomial Fourier transform $\mathcal{F} : \mathcal{P} \rightarrow F$ by

$$(\mathcal{F}p)(\gamma) = \langle p, E(\cdot, -\gamma) \rangle_{\mathbf{t}}, \quad p \in \mathcal{P}, \quad \gamma \in \text{Spec}(-Y).$$

Since the bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate, the map \mathcal{F} is an injective map. Moreover, $\mathcal{F}E(\cdot; \gamma_\lambda)$ is a multiple of δ_λ , hence \mathcal{F} maps an orthogonal basis of \mathcal{P} to an orthogonal basis of F , which implies that \mathcal{F} is surjective. We also define a linear map $\mathcal{G} : F \rightarrow \mathcal{P}$ by

$$(\mathcal{G}g)(x) = [g, E(x, -\cdot)]_{\mathbf{t}} = \sum_{\gamma \in \text{Spec}(-Y)} g(\gamma)E(x, -\gamma)N(-\gamma; \mathbf{t}), \quad g \in F, \quad x \in \mathbb{C}^n.$$

Theorem 5.12. *The map $\mathcal{G} : F \rightarrow \mathcal{P}$ is the inverse of the Fourier transform $\mathcal{F} : \mathcal{P} \rightarrow F$. Moreover, we have the Plancherel-type formulas*

$$[\mathcal{F}f_1, \mathcal{F}f_2]_{\mathbf{t}} = \langle f_1, f_2 \rangle_{\mathbf{t}}, \quad \langle \mathcal{G}g_1, \mathcal{G}g_2 \rangle_{\mathbf{t}} = [g_1, g_2]_{\mathbf{t}},$$

for $f_1, f_2 \in \mathcal{P}$ and $g_1, g_2 \in F$.

Proof. The proof is straightforward using the orthogonality relations for the nonsymmetric Wilson polynomials, see Theorems 5.8 and 5.11. \square

The affine Weyl group W has an action on F defined by

$$(wf)(-\gamma_\lambda) = f(-\gamma_{w^{-1}, \lambda}), \quad w \in W, f \in F.$$

We now define an action of \mathcal{H}_σ on F . For $f \in F$ and $Z \in \mathcal{H}_\sigma$ we set

$$(Zf)(\gamma) = (Z\bar{f})(\gamma). \quad (5.5)$$

Here \bar{f} denotes an arbitrary function $\bar{f} : \mathbb{C}^n \rightarrow \mathbb{C}$ such that $\bar{f}(\gamma) = f(\gamma)$ for any $\gamma \in \text{Spec}(-Y)$, and the action of \mathcal{H}_σ on the right hand side is given by the usual difference-reflection operators. We need to verify that (5.5) is well-defined.

Lemma 5.13. *The action of \mathcal{H}_σ on F defined by (5.5) is independent of the choice of extension \bar{f} of f .*

Proof. It is enough to verify that the lemma is true for generators T_i^σ , $i \in [0, n]$, and X_i , $i \in [1, n]$, of the algebra \mathcal{H}_σ . For the X_i 's this is obvious. Let $f \in F$ and let \bar{f} be an extensions of f . By Lemma 4.10 we have $(s_i \bar{f})(\gamma) = (s_i f)(\gamma)$ for $\gamma \in \text{Spec}(-Y)$ and $i \in [0, n]$. For the T_i^σ 's this gives us

$$(T_i^\sigma \bar{f})(\gamma) = \chi_i f(\gamma) + c_i^\sigma(\gamma) ((s_i f)(\gamma) - f(\gamma)),$$

for any extension \bar{f} of f . This is clearly independent of the choice of extension. \square

We have the following intertwining properties for the Fourier transforms \mathcal{F} and \mathcal{G} .

Proposition 5.14. *The Fourier transforms $\mathcal{F} : \mathcal{P} \rightarrow F$ and $\mathcal{G} : F \rightarrow \mathcal{P}$ have the following intertwining properties:*

$$\begin{aligned} \mathcal{F} \circ Z &= \sigma(Z) \circ \mathcal{F}, & Z \in \mathcal{H}, \\ \mathcal{G} \circ Z &= \sigma_\sigma(Z) \circ \mathcal{G}, & Z \in \mathcal{H}_\sigma. \end{aligned}$$

Proof. It is enough to check the intertwining property of \mathcal{F} for the generators Y_i, T_i, U_n , $i \in [1, n]$, of the algebra \mathcal{H} . This can be done using Lemma 5.7, Propositions 5.4 and 3.11.

The intertwining property of \mathcal{G} follows from the intertwining property of \mathcal{F} and Theorem 5.12. Let $g \in F$, then $g = \mathcal{F}f$ for a unique $f \in \mathcal{P}$, or equivalently $f = \mathcal{G}g$. Let $Z \in \mathcal{H}_\sigma$, then $Zg = \mathcal{F}(\sigma_\sigma(Z)f)$, which gives us

$$\mathcal{G}(Zg) = (\mathcal{G} \circ \mathcal{F})(\sigma_\sigma(Z)f) = \sigma_\sigma(Z)f = \sigma_\sigma(Z)\mathcal{G}g.$$

\square

6. Symmetric multivariable Wilson polynomials

In this section we define and study symmetric multivariable Wilson polynomials. They turn out to coincide with the multivariable Wilson polynomials as defined by van Diejen [21]. In [21] it is shown that the symmetric Wilson

polynomials in n variables diagonalize a system of n commuting difference operators, the duality property is obtained, and orthogonality relations are given. We derive these properties here from the representation theory of the rational DAHA \mathcal{H} , and we provide the link with the nonsymmetric theory.

6.1. Symmetric Wilson polynomials

For $\lambda \in \Lambda^+$ we define

$$\mathcal{P}(\lambda)^{W_0} = \mathcal{P}(\lambda) \cap \mathcal{P}^{W_0}.$$

Observe that it follows from $\mathcal{P} = \bigoplus_{\lambda \in \Lambda^+} \mathcal{P}(\lambda)$ that $\mathcal{P}^{W_0} = \bigoplus_{\lambda \in \Lambda^+} \mathcal{P}(\lambda)^{W_0}$.

Proposition 6.1. *For $\lambda \in \Lambda^+$ the space $\mathcal{P}(\lambda)^{W_0}$ is one-dimensional. Furthermore, any $f \in \mathcal{P}(\lambda)^{W_0}$ is given by*

$$f = \sum_{\mu \in W_0 \lambda} d_{\lambda, \mu} E(\cdot, \gamma_\mu), \quad (6.1)$$

with

$$d_{\lambda, \mu} = d_\lambda c_+^\sigma(-\gamma_\mu),$$

for some constant d_λ independent of $\mu \in W_0 \lambda$.

Proof. Let $i \in [1, n]$. From Proposition 5.4 we know that

$$(T_i - \chi_i)E(\cdot, \gamma_\mu) = c_{a_i}^\sigma(-\gamma_\mu) (E(\cdot, \gamma_{s_i \mu}) - E(\cdot, \gamma_\mu)),$$

if $s_i \mu \neq \mu$. Applying $(T_i - \chi_i)$ to the expansion (6.1) and using $(T_i - \chi_i)f = 0$ we find a recurrence relation for the coefficients $d_{\lambda, \mu}$;

$$d_{\lambda, s_i \mu} = \frac{c_{a_i}^\sigma(-\gamma_\mu)}{c_{-a_i}^\sigma(-\gamma_\mu)} d_{\lambda, \mu}. \quad (6.2)$$

Here we also used $c_\alpha(\gamma_{s_i \mu}) = (s_i c_\alpha)(\gamma_\mu) = c_{s_i \alpha}(\gamma_\mu)$. Let $s_{i_1} \cdots s_{i_r}$ be a reduced expression for v_μ (defined in the beginning of subsection 4.2), and define

$$\beta_1 = a_{i_1}, \quad \beta_j = s_{i_{j-1}} \cdots s_{i_1} a_{i_j}, \quad j = 2, \dots, r,$$

then iterating (6.2) gives

$$d_{\lambda, \mu} = d_{\lambda, \lambda} \prod_{j=1}^r \frac{c_{-\beta_j}^\sigma(\gamma_\lambda)}{c_{\beta_j}^\sigma(\gamma_\lambda)}.$$

From the definition of $c_+(x)$ we see that $(s_i c_+)(x) = c_+(x) c_{-a_i}(-x) / c_{a_i}(-x)$, and then

$$d_{\lambda, \mu} = d_{\lambda, \lambda} \frac{(v_\mu c_+^\sigma)(-\gamma_\lambda)}{c_+^\sigma(-\gamma_\lambda)} = d_{\lambda, \lambda} \frac{c_+^\sigma(-\gamma_\mu)}{c_+^\sigma(-\gamma_\lambda)}.$$

If we now set $d_{\lambda, \lambda} = d_\lambda c_+^\sigma(-\gamma_\lambda)$, the lemma follows. \square

In Sect. 6.5 we give an operator that maps $\mathcal{P}(\lambda)$ onto $\mathcal{P}(\lambda)^{W_0}$.

Definition 6.2. For $\lambda \in \Lambda^+$ we define the symmetric multivariable Wilson polynomial $E^+(x, \gamma_\lambda)$ to be the unique polynomial in $\mathcal{P}(\lambda)^{W_0}$ that takes the value 1 at $x = x_0$.

Since $f(x_0) = f(-x_0)$ if $f \in \mathcal{P}^{W_0}$ and $E(-x_0, \gamma_\lambda) = 1$, by Proposition 6.1 the symmetric Wilson polynomial $E^+(\cdot, \gamma_\lambda)$ is the polynomial in $\mathcal{P}(\lambda)^{W_0}$ with constant d_λ given by

$$d_\lambda = \left(\sum_{\mu \in W_0 \lambda} c_+^\sigma(-\gamma_\mu) \right)^{-1}.$$

This constant can be evaluated with the following result.

Lemma 6.3. *The function $K = \sum_{w \in W_0} w c_+$ is a constant function, and*

$$K(x) = \sum_{\mu \in W_0 \lambda} c_+(-x_\mu) = c_+(-x_0),$$

for any $\lambda \in \Lambda^+$.

Proof. We define the polynomial m by

$$m(x) = \prod_{\alpha \in \Sigma^-} \alpha(x) = (-2)^n \prod_{1 \leq j < k \leq n} (x_k^2 - x_j^2) \prod_{j=1}^n x_j.$$

This is an anti-symmetric polynomial, i.e., $(wm)(x) = (-1)^{l(w)}m(x)$ for any $w \in W_0$, where $l(w)$ denotes the length of w . It is well known that any anti-symmetric polynomial is the product of m with a symmetric polynomial. The product $m(x)K(x)$ is manifestly an anti-symmetric polynomial;

$$\begin{aligned} m(x)K(x) &= \sum_{w \in W_0} (-1)^{l(w)} w \\ &\times \left(\prod_{1 \leq j < k \leq n} (t + x_k - x_j)(t - x_k - x_j) \prod_{j=1}^n (a - x_j)(b - x_j) \right). \end{aligned}$$

Expanding both $m(x)$ and $m(x)K(x)$ in monomials x^μ , $\mu \in \mathbb{Z}_{\geq 0}^n$, we see that any monomial with nonzero coefficient in the expansion of $m(x)K(x)$ also has nonzero coefficient in the expansion of $m(x)$, hence $K(x)$ is a constant. Now let $\lambda \in \Lambda^+$, then by Lemma 4.10

$$K(x) = K(-x_\lambda) = \sum_{w \in W_0} (w c_+)(-x_\lambda) = \sum_{\substack{w \in W_0 \\ w\lambda \neq \lambda}} c_+(-x_{w\lambda}) + \sum_{\substack{w \in W_0 \\ w\lambda = \lambda}} (w c_+)(-x_\lambda).$$

Consider a term $(w c_+)(-x_\lambda)$ with $w \neq 1$ from the second sum. There exists a simple root a_i , $i \in [1, n]$, such that $\alpha = w^{-1}a_i \in \Sigma^-$. Then we see that the

factor $(wc_\alpha)(-x_\lambda) = c_{a_i}(-x_\lambda)$ equals zero, since $a_i(x_\lambda) = t$ for $i \in [1, n-1]$ and $a_n(x_\lambda) = 2t_n + 2u_n$. This shows that

$$K(x) = \sum_{\mu \in W_0 \lambda} c_+(-x_\mu).$$

In particular, for $\lambda = 0$ this gives $K(x) = c_+(-x_0)$. \square

6.2. Orthogonality relations

Next we show that the symmetric Wilson polynomials are orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathbf{t}}^+$. First a useful lemma. Let $V^+ \in W$ be the primitive idempotent

$$V^+ = \frac{1}{|W_0|} \sum_{w \in W_0} w,$$

where $|W_0| = 2^n n!$ is the cardinality of the finite Weyl group W_0 . Note that $V^+ f = f$ if $f \in \mathcal{P}^{W_0}$.

Lemma 6.4. *For $f, g \in \mathcal{P}^{W_0}$ we have*

$$\langle f, g \rangle_{\mathbf{t}} = \frac{c_+(-x_0)}{|W_0|} \langle f, g \rangle_{\mathbf{t}}^+.$$

Proof. First note that s_i , $i \in [1, n]$, is symmetric with respect to $\langle \cdot, \cdot \rangle_{\mathbf{t}}^+$, since $s_i \Delta^+ = \Delta^+$, hence so is V^+ . Let $f, g \in \mathcal{P}^{W_0}$, then by Lemma 6.3

$$\begin{aligned} \langle f, g \rangle_{\mathbf{t}} &= \langle V^+ f, g \rangle_{\mathbf{t}} = \langle V^+ f, c_+ g \rangle_{\mathbf{t}}^+ \\ &= \langle f, V^+(c_+ g) \rangle_{\mathbf{t}}^+ = \langle f, (V^+ c_+) g \rangle_{\mathbf{t}}^+ = \frac{c_+(-x_0)}{|W_0|} \langle f, g \rangle_{\mathbf{t}}^+. \end{aligned}$$

This proves the lemma. \square

Theorem 6.5. *The symmetric Wilson polynomials $E^+(\cdot, \gamma_\lambda)$, $\lambda \in \Lambda^+$, form an orthogonal basis for \mathcal{P}^{W_0} with respect to $\langle \cdot, \cdot \rangle_{\mathbf{t}}^+$.*

Proof. Since $\mathcal{P}^{W_0} = \bigoplus_{\lambda \in \Lambda^+} \mathcal{P}(\lambda)^{W_0}$ it is clear that the symmetric Wilson polynomials form a basis for \mathcal{P}^{W_0} . From the orthogonality relations for the nonsymmetric Wilson polynomials and the definition of the symmetric Wilson polynomials, it follows that

$$\langle E^+(\cdot, \gamma_\lambda), E^+(\cdot, \gamma_\mu) \rangle_{\mathbf{t}} = 0, \quad \lambda \neq \mu.$$

Now the orthogonality with respect $\langle \cdot, \cdot \rangle_{\mathbf{t}}^+$ follows from Lemma 6.4. \square

6.3. Difference equations

A difference-reflection operator $D \in \mathcal{P}_Y^{W_0}$ (considered as a subalgebra of $\text{End}(\mathcal{P})$) is of the form

$$D = \sum_{\lambda \in \Lambda, w \in W_0} c_{\lambda, w}(x) \tau(\lambda) w,$$

with $c_{\lambda,w}(x) \in \mathbb{C}(x)$. Restricted to the algebra of W_0 -invariant polynomials, it becomes a difference operator which we denote by D_{sym} ;

$$D_{sym} = \sum_{\lambda \in \Lambda, w \in W_0} c_{\lambda,w}(x) \tau(\lambda).$$

Moreover, D_{sym} is W_0 -invariant, i.e., $w \circ D_{sym} \circ w^{-1} = D_{sym}$ for any $w \in W_0$. Together with Theorem 4.17 this leads to the following property of the symmetric Wilson polynomials.

Theorem 6.6. *The symmetric Wilson polynomials satisfy the difference equation*

$$f(Y)_{sym} E^+(\cdot, \gamma_\lambda) = f(\gamma_\lambda) E^+(\cdot, \gamma_\lambda), \quad \lambda \in \Lambda^+,$$

for any $f \in \mathcal{P}^{W_0}$.

In particular, the symmetric Wilson polynomials are eigenfunctions of the following explicit difference operator.

Proposition 6.7. *The symmetric Wilson polynomials $E^+(\cdot, \gamma_\lambda)$, $\lambda \in \Lambda^+$, are eigenfunctions of the second order difference operator L defined by*

$$L = \sum_{i=1}^n A_i(x) (\tau(\epsilon_i) - 1) + A_i(-x) (\tau(-\epsilon_i) - 1),$$

$$A_i(x) = \frac{(a+x_i)(b+x_i)(c+x_i)(d+x_i)}{2x_i(2x_i+1)} \prod_{j \neq i} \frac{(t+x_i+x_j)(t+x_i-x_j)}{(x_i+x_j)(x_i-x_j)},$$

for eigenvalue $\sum_{i=1}^n \lambda_i(\lambda_i + a + b + c + d - 1 + 2(n-i)t)$.

Proof. Let $f_2 \in \mathcal{P}^{W_0}$ be defined by $f_2(x) = \sum_{i=1}^n x_i^2$. We show that $L = (f_2(Y) - f_2(\gamma_0))|_{sym}$. Then by Theorem 4.17 the symmetric Wilson polynomials $E^+(\cdot, \gamma_\lambda)$ are eigenfunctions of L for eigenvalue $\sum_i (\gamma_{\lambda,i}^2 - \gamma_{0,i}^2) = \sum_i \lambda_i(\lambda_i + 2\gamma_{\lambda,i})$, which is precisely the eigenvalue in the proposition.

Recall the reduced expression (2.1) for $\tau(\epsilon_i)$, and recall that $w\tau(\epsilon_i)w^{-1} = \tau(w\epsilon_i)$ for $w \in W_0$. From the explicit expression for the Y -operators we see that Y_i^2 expressed in terms of the generators T_j contains only terms in which T_0 occurs at most once (in the terms in which T_0 occurs twice, the T_0 's vanish using the quadratic relations in \mathcal{H}). Hence L is of the form

$$C(x) + \sum_{i=1}^n A_i(x) \tau(\epsilon_i) + B_i(x) \tau(-\epsilon_i)$$

with $A_i(x), B_i(x), C(x) \in \mathbb{C}(x)$. Since $Y_i 1 = \gamma_{0,i}$ and therefore $L1 = 0$, we see immediately that $C(x) = -\sum_i (A_i(x) + B_i(x))$. By the W_0 -invariance of L the coefficients A_i and B_i must satisfy

$$s_{\epsilon_i - \epsilon_j} A_i(x) = A_j(x), \quad s_{2\epsilon_i} A_i(x) = B_i(x).$$

Now it is enough to find the coefficient $A_1(x)$ of $\tau(\epsilon_1)$. We write

$$\sum_{i=1}^n Y_i^2 = \sum_{w \in W} c_w(x)w, \quad (6.3)$$

with $c_w(x) \in \mathbb{C}(x)$ and every $w \in W$ occurring in this expansion is an ordered subword of

$$s_i s_{i+1} \cdots s_n s_{n-1} \cdots s_0 s_1 \cdots s_{i-1} \quad \text{or} \quad s_{i-1} s_{i-2} \cdots s_0 s_1 \cdots s_n s_{n-1} \cdots s_i,$$

for $i \in [1, n]$. If we write here $s_0 = s_1 \cdots s_n \cdots s_1 \tau(\epsilon_1)$, and we use $u\tau(\epsilon_i) = \tau(u\epsilon_i)u$, $u \in W_0$, to write every w in (6.3) as $\tau(\pm\epsilon_i)v$ for some $v \in W_0$, then we see that $\tau(\epsilon_1)$ occurs only once. The only contribution to $\tau(\epsilon_1)$ comes from the expression $T_1 \cdots T_n \cdots T_0$ in Y_1^2 , and this gives us

$$A_1(x) = c_{\epsilon_1 - \epsilon_2}(x) \cdots c_{\epsilon_1 - \epsilon_n}(x) c_{2\epsilon_1}(x) c_{\epsilon_1 + \epsilon_n} \cdots c_{\epsilon_1 + \epsilon_2}(x) c_{\delta + 2\epsilon_1}(x).$$

Now the result follows from writing out this expression. \square

Remark 6.8. It can be shown that the difference operator L acts triangularly with respect to the dominance ordering on the W_0 -symmetric monomial $m_\lambda = \sum_{\mu \in S_n \lambda} x^{\phi(\mu)}$, $\lambda \in \Lambda^+$;

$$Lm_\lambda = \gamma_\lambda^+ m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu,$$

with γ_λ^+ the eigenvalue from Proposition 6.7. This shows that the eigenspace of L for eigenvalue γ_λ^+ is one-dimensional. The multivariable Wilson polynomials defined by van Diejen [20, 21], are eigenfunctions of L for eigenvalue γ_λ^+ , hence they coincide with our symmetric Wilson polynomials.

6.4. Duality

Using $f(Y)E^+(\cdot, \gamma_\lambda) = f(\gamma_\lambda)E^+(\cdot, \gamma_\lambda)$ for $f \in \mathcal{P}^{W_0}$, the duality property for the symmetric Wilson polynomials can be obtained in the same way as for the nonsymmetric ones, see Theorem 5.3.

Theorem 6.9. For $\lambda, \mu \in \Lambda^+$, we have

$$E^+(x_\mu, \gamma_\lambda) = E_\sigma^+(\gamma_\lambda, x_\mu).$$

Using the duality property, the difference equations from Theorem 6.6 give rise to recurrence relations for the symmetric Wilson polynomials.

6.5. The symmetrizer

We define an operator $C^+ : \mathbb{C}(x) \rightarrow \mathbb{C}(x)$ by

$$C^+ = V^+ c_+(X),$$

where $c_+(X)$ is multiplication by $c_+(x)$, see (5.3). We call C^+ the symmetrizer.

Lemma 6.10. The symmetrizer C^+ has the following properties:

- (i) For $f \in \mathcal{P}$ we have $C^+ f \in \mathcal{P}^{W_0}$.

- (ii) For $f \in \mathcal{P}^{W_0}$ we have $C^+f = |W_0|^{-1}c_+(-x_0)f$.
 (iii) For $i \in [1, n]$, $C^+T_i = \chi_i C^+ = T_i C^+$.

Proof. (i) Let us denote $|W_0|(C^+f)(x) = K_f(x)$. In the same way as in the proof of Lemma 6.3 we find that the product $m(x)K_f(x)$ is an anti-symmetric polynomial, from which it follows that $K_f \in \mathcal{P}^{W_0}$.

(ii) If $f \in \mathcal{P}^{W_0}$ we see that $K_f = fK_1$. Then the result follows from Lemma 6.3.

(iii) Let $i \in [1, n]$. Using $T_i = \chi_i + c_i(X)(s_i - 1)$ and $c_{a_i} + c_{-a_i} = 2\chi_i$ we find $T_i + \chi_i = (s_i + 1)c_{-a_i}(X)$, since $s_i c_{a_i} = c_{s_i a_i}$. Furthermore, from the definition of $c_+(x)$ it follows that $s_i c_+(X)c_{a_i}(X) = c_+(X)s_i c_{-a_i}(X)$. This gives

$$\begin{aligned} C^+(T_i + \chi_i) &= V^+c_+(X)s_i c_{-a_i}(X) + V^+c_+(X)c_{-a_i}(X) \\ &= V^+s_i c_+(X)c_{a_i}(X) + V^+c_+(X)c_{-a_i}(X) \\ &= V^+c_+(X)2\chi_i \\ &= 2\chi_i C^+, \end{aligned}$$

since $V^+s_i = V^+$. This proves the identity $C^+T_i = \chi_i C^+$. The identity $T_i C^+ = \chi_i C^+$ follows from (i), since $T_i f = \chi_i f$ for $f \in \mathcal{P}^{W_0}$. \square

We show that the symmetrizer C^+ maps $\mathcal{P}(\lambda)$ onto $\mathcal{P}(\lambda)^{W_0}$ for all $\lambda \in \Lambda^+$.

Proposition 6.11. For $\lambda \in \Lambda^+$ and $\mu \in W_0\lambda$, we have

$$C^+E(\cdot, \gamma_\mu) = \frac{c_+(-x_0)}{|W_0|}E^+(\cdot, \gamma_\lambda).$$

Proof. Let $\lambda \in \Lambda^+$ and $\mu \in W_0\lambda$. We know that $C^+E(\cdot, \gamma_\mu) \in \mathcal{P}^{W_0}$, so by Theorem 6.5 there exists an expansion

$$C^+E(\cdot, \gamma_\mu) = \sum_{\nu \in \Lambda^+} b_{\mu, \nu} E^+(\cdot, \gamma_\nu).$$

Using the orthogonality we find

$$\begin{aligned} b_{\mu, \nu} \langle E^+(\cdot, \gamma_\nu), E^+(\cdot, \gamma_\mu) \rangle_{\mathbf{t}}^+ &= \langle E^+(\cdot, \gamma_\nu), C^+E(\cdot, \gamma_\mu) \rangle_{\mathbf{t}}^+ \\ &= \langle V^+E^+(\cdot, \gamma_\nu), c_+(x)E(\cdot, \gamma_\mu) \rangle_{\mathbf{t}}^+ \\ &= \langle E^+(\cdot, \gamma_\nu), E(\cdot, \gamma_\mu) \rangle_{\mathbf{t}}. \end{aligned} \quad (6.4)$$

Since $E^+(\cdot, \gamma_\nu) \in \mathcal{P}(\nu)^{W_0}$ we obtain from the orthogonality relations for the nonsymmetric Wilson polynomials that $b_{\mu, \nu} = 0$ if $\mu \notin W_0\nu$, i.e., if $\nu \neq \lambda$. We can find $b_{\mu, \lambda}$ by evaluating at x_0 . Using $(wc_+)(-x_0) = 0$ if $w \neq 1$, we obtain

$$\begin{aligned} (C^+E(\cdot, \gamma_\mu))(x_0) &= |W_0|^{-1} \sum_{w \in W_0} (wc_+)(-x_0) (wE(\cdot, \gamma_\mu))(-x_0) \\ &= |W_0|^{-1} c_+(-x_0). \end{aligned}$$

This proves the proposition. \square

6.6. Quadratic norms

Finally, we derive the quadratic norms for the symmetric Wilson polynomials, see [21, Theorem 7.4].

Theorem 6.12. *For $\lambda \in \Lambda^+$,*

$$\langle E^+(\cdot, \gamma_\lambda), E^+(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}}^+ = \frac{1}{N_+(\gamma_\lambda)},$$

where

$$N_+(\gamma_\lambda) = \frac{1}{\langle 1, 1 \rangle_{\mathbf{t}}^+} \frac{c_+^\sigma(-\gamma_\lambda)}{c_+^\sigma(-\gamma_0)} \prod_{\alpha \in \mathcal{R}_r^+ \cap \tau(\lambda) \mathcal{R}_r^-} \frac{c_{-\alpha}^\sigma(\gamma_\lambda)}{c_\alpha^\sigma(\gamma_\lambda)}$$

Proof. From (6.4) and Theorem 5.11 we obtain

$$\begin{aligned} \langle E^+(\cdot, \gamma_\lambda), E^+(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}}^+ &= \frac{|W_0|}{c_+(-x_0)} \langle E^+(\cdot, \gamma_\lambda), E(\cdot, \gamma_\lambda) \rangle_{\mathbf{t}} \\ &= \frac{|W_0| d_{\lambda, \lambda}}{c_+(-x_0) N(\gamma_\lambda)}, \end{aligned}$$

where $d_{\lambda, \lambda}$ is the coefficient of $E(\cdot, \gamma_\lambda)$ in the expansion of $E^+(\cdot, \gamma_\lambda)$ from Proposition 6.1, i.e.,

$$d_{\lambda, \lambda} = \frac{c_+^\sigma(-\gamma_\lambda)}{c_+^\sigma(-\gamma_0)}.$$

Writing out $N(\gamma_\lambda)$ and using Lemma 6.4 gives the result. \square

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